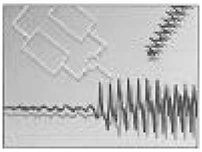
Signals and Systems



Signals and Systems

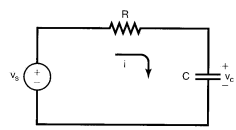
- ***Signals*** are variables that carry information
- ***Systems*** take signals as inputs and produce signals as outputs

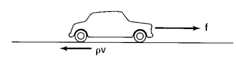
The course deals with the passage of signals through systems

Signals

- ***Signals*** describe a wide variety of (physical) phenomena
- ***Signals*** may be represented in many ways
- ***Information*** in a signal is contained in a pattern of variations of some form, i.e.
  - variation of voltages over time in a circuit
  - applied force and resulting velocity of a car
  - fluctuations of acoustic pressure in speech production by human vocal mechanism

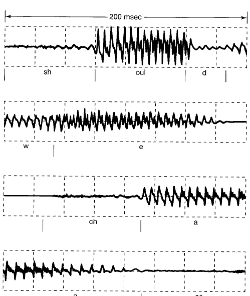
Examples of Signals





- A simple RC circuit with source voltage v_s and capacitor voltage v_c
- An automobile responding to an applied force f from the engine and to a frictional force ρv proportional to the velocity v

Examples of Signals



Example of a recording of speech: The signal represents acoustic pressure variations as a function of time for the spoken words: *"should we chase"*

Examples of Signals

A monochromatic picture

Representation of Signals

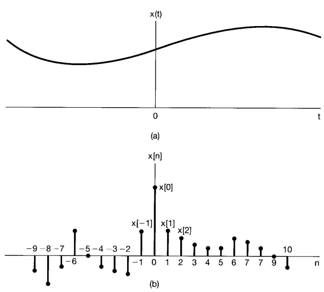
- Signals are represented mathematically as functions of one or more independent variables
- We will generally refer the independent variable as time
- Two basic types of signals:
  - Continuous-time (CT) signals and
  - Discrete-time (DT) signals

Continuous-Time and Discrete-Time Signals

- Symbol t is used to denote the independent variable of continuous-time signals
- Symbol n is used to denote the independent variable of discrete-time signals
- Continuous-time signal: $x(t)$
- Discrete-time signal: $x[n]$

  $x[n]$ is a ***sequence***, defined only for integer values of n

Continuous-Time and Discrete-Time Signals



Digital Image

- Two-dimensional (digital) signal: Intensity is a function of spatial coordinates

Signal Energy and Power

- Signals are directly related to physical quantities capturing power and energy in a physical system
- ***Instantaneous power***, e.g.,

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

where $v(t)$ and $i(t)$ are the voltage and current, respectively, across the resistor of resistance R

Signal Energy and Power

- Total ***energy*** expended over the time interval $t_1 \le t \le t_2$

$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

- ***Average power*** over this time interval

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

Signals with Complex Values

- Total energy over time interval $t_1 \le t \le t_2$ of a continuous-time signal $x(t)$

$$\int_{t_1}^{t_2} |x(t)|^2 \, dt$$

where $|x|$ is the magnitude of the (possibly complex) number x

- Similarly, the total energy of a discrete-time signal $x[n]$ over time interval $t_1 \le t \le t_2$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

Total Energy over Infinite Time Interval

- In many systems we are interested in examining power and energy in signals over infinite time interval

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

and

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Total Power over Infinite Time Interval

- The time-averaged power over infinite time interval is defined as

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

and

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

Three Important Classes of Signals 1(3)

- Signals with finite total energy, E_∞
Such a signal must have zero average power

$$P_\infty = \lim_{t\to\infty} \frac{E_\infty}{2T} = 0$$

- Example: A signal that takes the value 1 for $0 \le t \le 1$ and 0 otherwise.
In this case

$$E_\infty = 1 \text{ and } P_\infty = 0$$

Three Important Classes of Signals 2(3)

- Signals with finite average power P_∞
- If $P_\infty > 0$ then, of necessity, $E_\infty = \infty$
- If there is nonzero average energy per unit time (i.e. nonzero power) , then integrating or summing this over an infinite time interval yields an infinite amount of energy
- Example: Constant signal $x[n] = 4$ has infinite energy, but average power $P_\infty = 16$

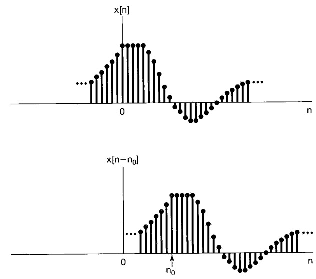
Three Important Classes of Signals 3(3)

- There are also signals for which neither P_∞ nor E_∞ are infinite
- A simple example is the signal $x(t) = t$

Transformations of the Independent Variable

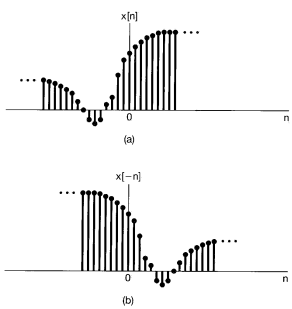
- Time shift: $x[n] = x[n-n_0]$
- Time reversal: $x[-n]$ obtained from $x[n]$
- Time scaling: $x(t)$; $x(2t)$; $x(t/2)$
- Transformation: $x(t) \rightarrow x(at + b)$ preserves the shape of $x(t)$;
  - linear stretching if $|a| < 1$ or
  - linear compression if $|a| > 1$
  - time reversal if $a < 0$
  - time shift if b is nonzero

Time-Shift of a Discrete-Time Signal

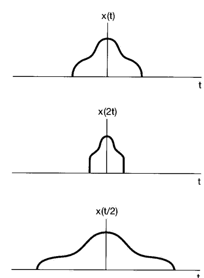


- Original sequence $x[n]$
- Delayed sequence $x[n-n_0]$

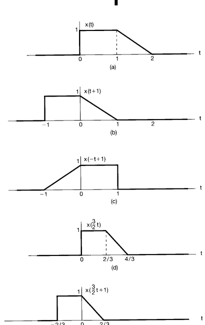
Time-Reversal of a Discrete-Time Signal



Time-Scaling of a Continuous-Time Signal



Examples of Operations



Signal $x(t)$

Advance $x(t+1)$ (shift to the left)

Reversed version of $x(t+1)$: $x(-t+1)$

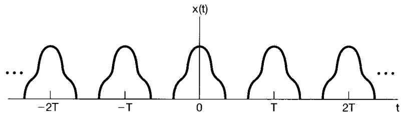
Compressed version of $x(t)$: $x((3/2)t)$

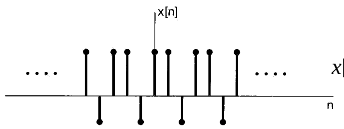
Linearly compressed and advanced signal: $x((3/2)t+1)$

Periodic Signals

- A signal $x(t)$ is ***periodic with period T*** if
  $x(t) = x(t+T)$
  for all values of t
- The ***fundamental period T_0*** of $x(t)$ is the smallest positive value of T for which the above equality holds
- A signal $x(t)$ that is not periodic is referred to as an ***aperiodic*** signal

Examples of Periodic Signals





$x[n] = x[n+3]$

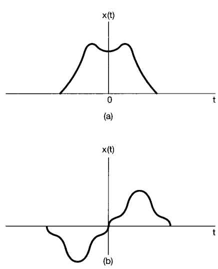
Even and Odd Signals

- Continuous and discrete even signals:

  $x(-t) = x(t)$ or $x[-n] = x[n]$
- Continuous and discrete odd signals:

  $x(-t) = -x(t)$ or $x[-n] = -x[n]$
- An odd signal must be necessarily zero at $t = 0$ or $n = 0$

Examples of Even and Odd Signals



Even-Odd Decomposition of a Signal

- Even part of $x(t)$:

  Ev $\{x(t)\} = [x(t)+x(-t)] / 2$
- Odd part of $x(t)$:

  Od $\{x(t)\} = [x(t)-x(-t)] / 2$

Continuous-Time Complex Exponential and Sinusoidal Signals

- Complex exponential signal: $x(t) = C\, e^{at}$
  where C and a are in general complex numbers
- Real exponential signals: C and a are real

Growing exponential: $a>0$ Decaying exponential: $a<0$

Periodic Complex Exponential and Sinusoidal Signals

Number a is purely imaginary: $x(t) = e^{jw_0 t}$

$x(t)$ is periodic with period T: $e^{jw_0 t} = e^{jw_0 (t+T)}$

Since $e^{jw_0 (t+T)} = e^{jw_0 t} e^{jw_0 T}$

It follows that for periodicity, we must have $e^{jw_0 T} = 1$

If w_0=0 the $x(t)$=1 which is periodic for any value of T.
If w_0 is nonzero, then the ***fundamental period*** T_0 of $x(t)$ is

$$T_0 = \frac{2p}{|w_0|}$$

Periodic Complex Exponential and Sinusoidal Signals

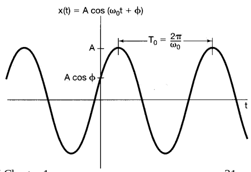
- A signal closely related to the periodic complex exponential is the ***sinusoidal signal***

$$x(t) = A\cos(w_0 t + f)$$

- It is common to write $w_0 = 2pf_0$

f_0 has units of cycles per second or Hertz (Hz)

w_0 has units of radians per second

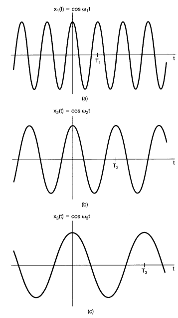


Sinusoidal Signal

- Fundamental period: T_0
- Fundamental frequency: w_0=1/T_0

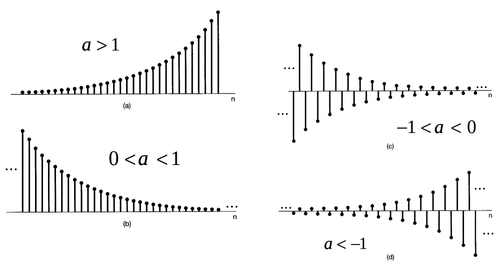
- Illustration:

$$T_1 < T_2 < T_3$$
$$w_1 > w_2 > w_3$$

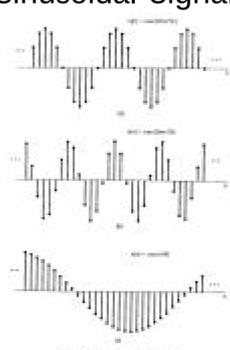


Discrete-Time Complex Exponential Signals

$$x[n] = Ca^n = Ce^{bn}; \quad a = e^b$$



Sinusoidal Signals (Sequences)



$$x[n] = \cos(2pn/12)$$

$$x[n] = \cos(8pn/31)$$

$$x[n] = \cos(n/6)$$

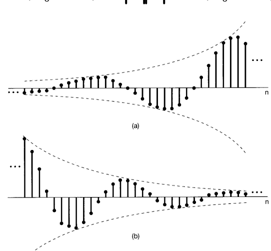
General Complex Exponential Signals

$$C = |C|e^{jq}, \quad a = |a|e^{jw_0}$$

$$Ca^n = |C||a|^n \cos(w_0 n + q) + j|C||a|^n \sin(w_0 n + q)$$

$|a| > 1$

$|a| < 1$



Periodicity Properties of Discrete-Time Complex Exponentials

- Continuous-time $\exp(jw_0 t)$:
  1. Increasing w_0 increases the rate of oscillation
  2. $\exp(jw_0 t)$ is periodic for any value of w_0
- Consider DT complex exponential with frequency w_0+2p:

$$e^{j(w_0+2p)n} = e^{jw_0 n}e^{j2pn} = e^{jw_0 n}$$

- The exponential at frequency w_0+2p is the same as that of frequency w_0
- In CT case, the exponential signals $\exp(jw_0 t)$ are all distinct for distinct values of w_0

Periodicity Properties of Discrete-Time Complex Exponentials

- In DT case, the signals are not distinct, as the signal with frequency w_0 is identical to the signals to the signals with frequencies $w_0 \pm 2p$, $w_0 \pm 4p$ etc.
- Considering complex exponentials we need only consider a frequency interval of length $2p$, i.e.,

$$0 \le w_0 < 2p \quad \text{or} \quad -p \le w_0 < p$$

DT Sinusoidal Sequences

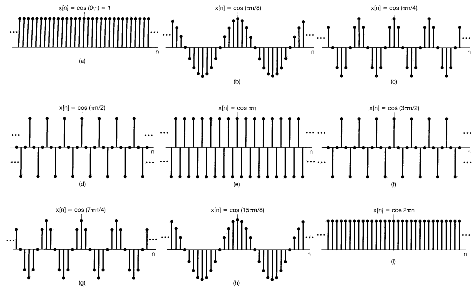


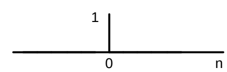
Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

The Unit Impulse and the Unit Step Functions

Some Basic Sequences

- ***Unit sample sequence***

$$d[n] = \begin{cases} 1, & n = 0 \\ 0, & n \ne 0 \end{cases}$$



- ***Unit step sequence***

$$m[n] = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

Relations between Basic Sequences

- Unit sample and unit step sequences are related as follows:

$$d[n] = m[n] - m[n-1]$$

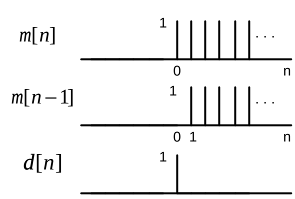
$$m[n] = \sum_{m=-\infty}^{n} d[m]$$

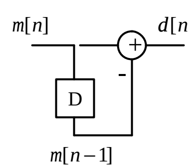
- The above relations can be implemented with simple computational structures consisting of basic arithmetic operations

Relations between Basic Sequences

- The unit sample is the first difference of the unit step:

$$d[n] = m[n] - m[n-1]$$



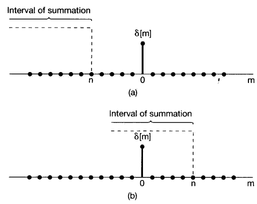


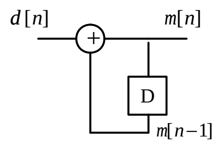
Realization

Relations between Basic Sequences

- Unit step is the running sum of the unit step:

$$m[n] = \sum_{m=-\infty}^{n} d[m] = \sum_{m=-\infty}^{n-1} d[m] + d[n] = m[n-1] + d[n]$$





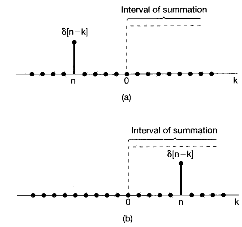
Realization

Relations between Basic Sequences

- By changing the variable of summation in the running sum from m to $k=n-m$, the discrete-time unit step can be written in terms of the unit sample as

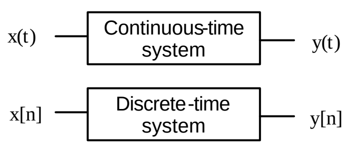
$$m[n] = \sum_{k=\infty}^{0} d[n-k]$$

$$= \sum_{k=0}^{\infty} d[n-k]$$



Continuous-Time and Discrete-Time Systems

- A system can be viewed as a process in which input signals are transformed by the system resulting in other signals as outputs



Examples

- Example 1.8: An RC circuit
- Example 1.9: A forces affecting the car
- Example 1.10: A balance in a bank account
- Example 1.11: Digital simulation of the differential equation

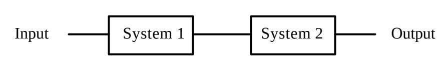
Mathematical Descriptions of Systems

- Classes of systems that have two important characteristics:
  1) The systems have properties and structures that can be exploited to gain insight into their behavior and to develop effective tools for their analysis
  2) Many systems of practical importance can be accurately modeled using these systems

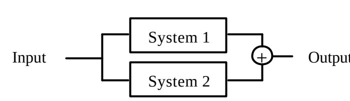
  Tools are developed for a particular class of systems referred to as
  linear and time-invariant systems

Interconnections of Systems

- Series or cascade interconnection

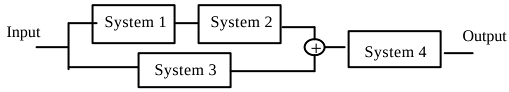


- Parallel interconnection

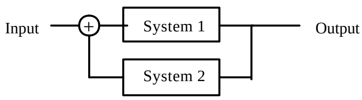


Interconnections of Systems

- Combination of parallel and cascade interconnections



- Feedback interconnection



Basic System Properties

- Systems with and without memory
- Invertibility and inverse systems
- Causality
- Stability
- Time invariance
- Linearity
- Convolution

Memoryless Systems

- Output for each value of the independent variable at a given time is dependent only on the input at the same time

$$Example: \quad y[n] = \left(2x[n] - x^2[n]\right)^2$$

Identity System

$$Continuous-time: \quad y(t) = x(t)$$

$$Discrete-time: \quad y[n] = x[n]$$

- An identity system is a simple memoryless system whose output is identical to its input

Accumulator

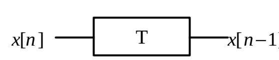
$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- An accumulator is a discrete-time system with memory

Delay

$$y[n] = x[n-1]$$

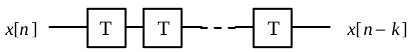
- The output is the delayed version of the input
- Realization using a memory location or register with delay T



Arbitrary Delay

$$y[n] = x[n-k]$$

- An arbitrary delay of k time instants can be realized using a shift register of length k



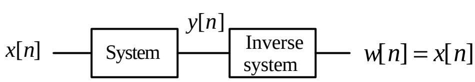
Accumulator or Running Sum

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n]$$

$$y[n] = y[n-1] + x[n]$$

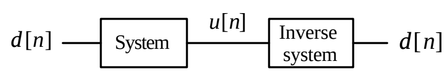
- The accumulator must remember the running sum of previous input values to obtain the output at current time n

Invertible Systems



- If a system is invertible, then an inverse system exists that when cascaded with the original system yields an output w[n] equal to input x[n]

Invertible Systems



$$u[n] = \sum_{k=-\infty}^{n} d[k] \qquad d[n] = u[n] - u[n-1]$$

- Accumulator is an invertible discrete-time system

Causality

- A system is ***causal*** if the output at any time depends only on the values of the input at the same time and in the past
- *Example:*
  Accumulator and delay are causal systems
- All memoryless systems are causal

Noncausality

- A system is ***noncausal*** if the output at any time depends also on the future values of the input
- Noncausal systems are physically not realizable

$$\textit{Example}: \qquad y[n] = x[n] - x[n+1]$$

Noncausality

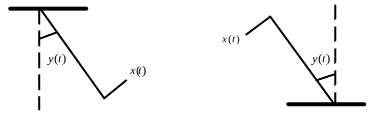
- A noncausal averaging filter

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$$

- The filter can be realized with a delay of M samples

Stability

- Informally, a system is stable if small inputs lead to responses that do not diverge



Pendulum *Inverted pendulum*

Time Invariance

- A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal

$$y[n] = T\left(x[n]\right)$$

$$y[n-n_0] = T\left(x[n-n_0]\right)$$

- The system properties do not change with time

Time Invariance

- A ***time invariant*** continuous-time system

$$y(t) = \sin\left[x(t)\right]$$

- A ***time variant*** discrete-time system

$$y[n] = nx[n]$$

Coefficient n is changing with time

Linearity

- A ***linear system*** is a system that possesses the important property of superposition

Additivity:

The response to $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$

Scaling or homogeneity:

The response to $ax_1(t)$ is $ay_1(t)$

where a is any complex constant

Linearity

- Combining the two properties of superposition into a single statement

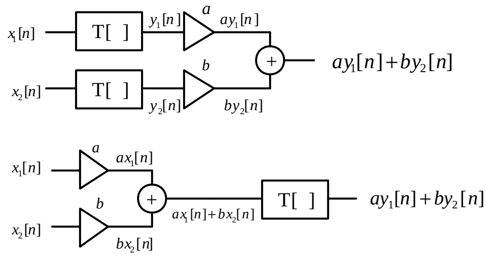
Discrete-time:

$$ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

where a and b are any complex constants

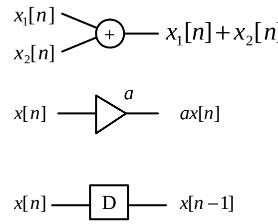
The superposition property holds for linear systems in continuous and discrete time

Linearity

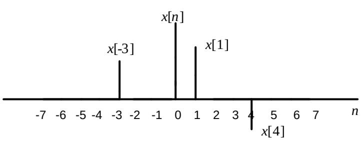


Basic Operations on Sequences

- Addition:
- Multiplication:
- Unit delay:

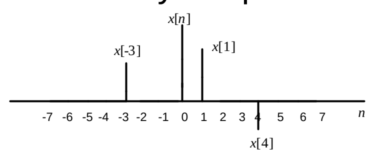


Arbitrary Sequence



- An arbitrary sequence $x[n]$ can be expressed as a superposition of scaled versions of shifted unit impulses, $d[n\text{-}k]$

Arbitrary Sequence



$$x[n] = \boxed{x[-3]d[n+3]} + \boxed{x[1]d[n-1]} - \boxed{x[4]d[n-4]}$$

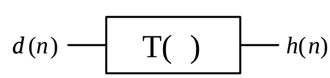
- In general: $x[n] = \sum_{k=-\infty}^{+\infty} x[k]d[n-k]$

Convolution

- x[n] is represented as a superposition of scaled versions of shifted unit impulses, δ[n-k]
- ***Linearity:*** The response of a linear system to x[n] will be the superposition of the scaled responses of the system to each of these shifted impulses
- ***Time invariance:*** The responses of a time-invariant system to time-shifted unit impulses are the time-shifted versions of one another

Convolution

- The unit impulse response of a system is $h[n]$



Convolution

$$y[n] = T(x[n]) = T\left(\sum_{k=-\infty}^{\infty} x[k]d[n-k]\right)$$

Additivity: $\quad y[n] = \sum_{k=-\infty}^{\infty} T(x[k]d[n-k])$

Homogeneity: $\quad y[n] = \sum_{k=-\infty}^{\infty} x[k]T(d[n-k])$

Shift – invariance: $\quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$