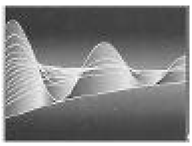
Time and Frequency Characterization of Signals and Systems



Time and Frequency Characterization of Signals and Systems

- **Frequency-domain characterization of an LTI system in terms of its frequency response represents an alternative to the time-domain characterization through convolution**
- **In system design and analysis, it is important to relate time-domain and frequency-domain characteristics and trade-offs**

The Magnitude and Phase Representation of the Fourier Transform

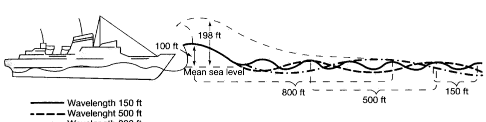
- The Fourier transform is complexvalued and its real and imaginary parts can be represented in terms of ***magnitude*** and ***phase***
- In continuous-time $X(jw) = |X(jw)|\, e^{j \arg[X(jw)]}$
- In discrete-time $X(e^{jw}) = |X(e^{jw})|\, e^{j \arg[X(e^{jw})]}$
- From the synthesis equation (in CT) $x(t) = \frac{1}{2p} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$
- $X(jw)$ provides a decomposition of the signal $x(t)$ into a "sum" of complex exponentioals at different freuencies

The Magnitude and Phase Representation of the Fourier Transform

- ***The magnitude*** $|X(jw)|$ describes the basic ***frequency content*** of the signal
- The magnitude $|X(jw)|$ provides the information about the relative magnitudes of the complex exponentials that make the signal $x(t)$
- ***The phase angle*** $\arg[X(jw)]$ does not affect the amplitudes of the individual frequency components, but instead provides information concerning the relative phases of the exponentials
- The phase relationships captured by $\arg[X(jw)]$ have a significant effect on the nature of the signal $x(t)$ and thus contain a substantial amount of information about the signal

Example

- A ship encounters the superposition of three wave trains, each of which can be modeled as a sinusoidal signal



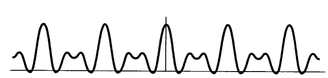
- With fixed magnitudes for these sinusoids, the amplitude of their sum may be quite small or very large depending on their relative phases

Example: Linear combination of sinusoidal signals

Consider the signal: $x(t) = 1 + \frac{1}{2}\cos 2pt + \cos 4pt + \frac{2}{3}\cos 6pt$

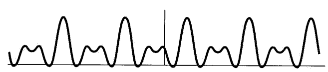
The same sinusoidal components with phase shifts:

$$x(t) = 1 + \frac{1}{2}\cos(2pt + f_1) + \cos(4pt + f_2) + \frac{2}{3}\cos(6pt + f_3)$$



(a)

(a) $F_1 = F_2 = F_3 = 0$



(b)

(b) $F_1 = 4$, $F_2 = 8$, and $F_3 = 12$ rad

Example: Linear combination of sinusoidal signals

$$x(t) = 1 + \frac{1}{2}\cos(2pt + f_1) + \cos(4pt + f_2) + \frac{2}{3}\cos(6pt + f_3)$$

(c)

(c) $F_1 = 6$, $F_2 = -2.7$, $F_3 = 0.93$ rad

(d)

(d) $F_1 = 1.2$, $F_2 = 4.1$, $F_3 = -7.02$ rad

The resulting signals differ significantly for different relative phases

The Magnitude and Phase Representation of the Fourier Transform

- Changes in the phase function of $X(jw)$ lead to changes in the time-domain characteristics of the signal $x(t)$
- In some applications, ***the phase distortion*** may be important, whereas in others it is not
- For example, the auditory system is relatively insensitive to phase
  - Mild phase distortion such as those affecting individual sounds do not lead to a loss of intelligibility
  - More severe distortion of speech, however, do affect
- Example: $x(t)$ is a tape recording of a sentence; $x(-t)$ is the same sentence played backward. Assuming that $x(t)$ is real valued

$$F\{x(-t)\} = X(-jw) = |X(jw)|\, e^{-j\arg[X(jw)]}$$

i.e. the spectrum has the same magnitude function and differs only in phase => ***The phase does affect intelligibility***

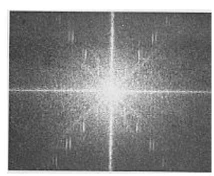
The Magnitude and Phase Representation of the Fourier Transform

- The importance of phase is found in examining images:
  - A monochrome image can be thought of as a signal $x(t_1,t_2)$ with t_1 and t_2 denoting the spatial variables, i.e., horizontal and vertical coordinate points, respectively, and $x(t_1,t_2)$ the brightness of the image at the point (t_1,t_2)
  - The Fourier transform $X(jw_1, jw_2)$ of the image represents a decomposition of the image into complex exponential components of the form $e^{jw_1t_1}e^{jw_2t_2}$ that capture the spatial variations of $x(t_1,t_2)$ at different frequencies in each of the two coordinate directions
- In images, most important visual information is in the edges and regions of high contrast
- Intuitively, maximum and minimum intensity in the image are places at which complex exponentials at different frequencies are in phase

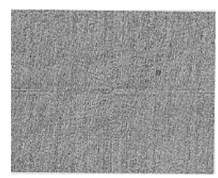
=> ***The phase should capture the information about the edges***

Example: Two-Dimensional Image

(a) Monochrome image $x(t_1,t_2)$

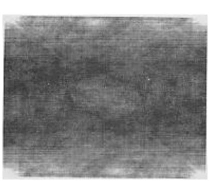


(b) Magnitude of $X(jw_1, jw_2)$

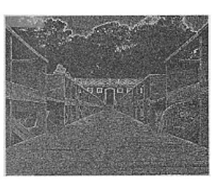


(c) Phase of $X(jw_1, jw_2)$

Example: Two-Dimensional Image

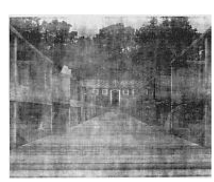


(d) Inverse Fourier transform of $X(jw_1, jw_2)$ with setting the phase to zero



(e) Inverse Fourier transform with $|X(jw_1, jw_2)|$ set equal to 1 and keeping the phase unchanged

Example: Two-Dimensional Image



(f) Inverse Fourier transform of a mixture with the original phase and the magnitude of a completely different image

(g) The image the magnitude of which has been used in (f)

The Magnitude and Phase Representation of the Frequency Response of LTI Systems

$x[n]$ → $h[n]$ → $y[n] = x[n]*h[n]$

In frequency domain:

$$Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

The effect that an LTI system has on the input is to change the complex amplitude of each of the frequency components of the signal

The Magnitude and Phase Representation of the Frequency Response of LTI Systems

- In terms of the magnitude-phase representation, the nature of the effect can be understood in more detail

$$|Y(e^{jw})| = |H(e^{jw})|\,|X(e^{jw})|$$

$$\arg[Y(e^{jw})] = \arg[H(e^{jw})] + \arg[X(e^{jw})]$$

- $|H(e^{jw})|$ is the ***gain*** of the system
- $\arg[H(e^{jw})]$ is the ***phase shift*** of the system

Linear and Nonlinear Phase

- When the phase shift in w is a linear function of w, there is a straightforward interpretation of the effect in the time domain
- Consider the DT LTI system with frequency response

$$H(e^{jw}) = e^{-jwn_0}$$

so the system has unit gain and and linear phase, i.e.,

$$|H(e^{jw})| = 1;\quad \arg[H(e^{jw})] = -w\,n_0$$

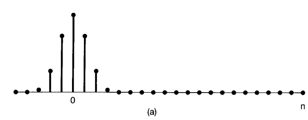
- This corresponds to a time shift in the time domain, i.e.

$$y[n] = x[n-n_0]$$

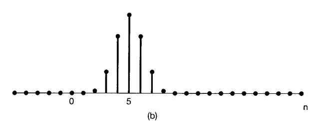
- The linear phase property means that all the components of the signal (complex exponentials) are delayed in the system with n_0

The Effect of Linear and Nonlinear Phase

- Input is applied to three different systems with unit gain ($|H(e^{jw})|=1$)



(a) Input signal

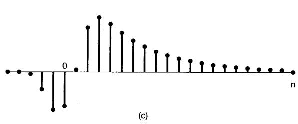


(b) Linear phase

$$H_1(e^{jw}) = e^{-jw5}$$

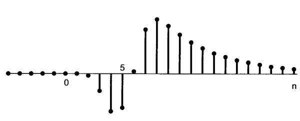
i.e., the input is delayed by 5 samples

The Effect of Linear and Nonlinear Phase



(c) Nonlinear phase

$$H_2(e^{jw})$$



(d) Nonlinear phase and delay

$$H_3(e^{jw}) = H_2(e^{jw})H_1(e^{jw})$$

- The systems with $|H(e^{jw})|=1$ are referred to as ***allpass systems***

Group Delay

- The phase slope of linear phase systems tells us the size of the time shift
- $\arg[H(e^{jw})] = -wn_0$ corresponds to a shift or delay of n_0 samples
- Extending the concept of delay to nonlinear phase characteristics, consider effects of the phase of a DT LTI system on a ***narrowband*** input signal $x[n]$, i.e., $X(e^{jw})$ is zero or negligibly small outside a small band of frequencies centered at $w=w_0$
- We can approximate the phase within this band by a linear approximation (first-order approximation)

$$\arg[H(e^{jw})] \approx -f - w\,n_0$$

so that

$$Y(e^{jw}) \cong X(e^{jw})\,|H(e^{jw})|\,e^{-jf}e^{-jwn_0}$$

Group Delay

$$Y(e^{jw}) \cong X(e^{jw})\,\left|H(e^{jw})\right| e^{-jf} e^{-jwn_0}$$

- The magnitude shaping of the narrowband input corresponds to $|H(e^{jw})|$ and the phase shaping with multiplication by an overall complex factor e^{-jF} and multiplication by a linear phase term e^{-jn0} corresponding to a time shift of n_0 samples

  The time shift (delay) n_0 is referred to as the group delay at $w=w_0$
- It is the ***effective common delay*** experienced by the small band or group of frequencies centered at w_0
- The ***group delay*** at each frequency equals the negative of the slope of the phase at that specific frequency, i.e., the group delay is defined as

$$t(w) = -\frac{d}{dw}\left\{\arg\{X(e^{jw})\}\right\}$$

Example 6.1: A CT all-pass system with varying group delay

$$H(jw) = \prod_{i=1}^{3} H_i(jw), \quad \text{where} \quad H_i(jw) = \frac{1+(jw/w_i)^2 - 2jz_i(w/w_i)}{1+(jw/w_i)^2 + 2jz_i(w/w_i)}$$

Now, $w_i = 2pf_i$ with $f_1 \cong 50$ Hz, $f_2 \cong 150$ Hz, and $f_3 \cong 300$ Hz

$$\text{Magnitude}: \ |H(jw)| = 1; \quad \text{Phase}: \ \arg[H(jw)] = \sum_{i=1}^{3} \arg[H_i(jw)]$$

$$\text{Group delay}: \quad t(w) = -\frac{d}{dw}\left\{\sum_{i=1}^{3} \arg[H_i(jw)]\right\}$$

- ***Principal phase*** with values between $-p$ and p (i.e. the phase modulo $2p$)
- Non-constant group delay causes dispersion in the impulse response
- $F\{d(t)\}=1$ => All frequencies of the impulse are aligned in time in such a way that they combine to form an impulse

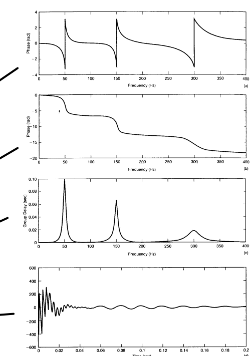
Example 6.1: A CT all-pass system with varying group delay

a) Principal phase

b) Unwrapped phase

c) Group delay

d) Impulse response



Log-Magnitude and Bode Plots

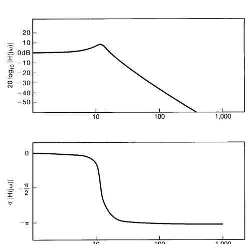
- Logarithmic scale for the magnitude of the Fourier transform is often convenient both in CT and DT
- The product of magnitudes in CT can be displayed in logarithmic scale as additive relationship, i.e.

$$|Y(jw)| = |H(jw)|\,|X(jw)| \quad \Rightarrow \quad \log|Y(jw)| = \log|H(jw)| + \log|X(jw)|$$

- Consequently, the graph of the Fourier transform of the output of the system is obtained by adding the log-magnitudes of $H(jw)$ and $X(jw)$ and the phases of $H(jw)$ and $X(jw)$, i.e., the plots can be added to obtain the graphical representation
- In addition, the logarithmic scale allows details to be displayed over a wider dynamic range

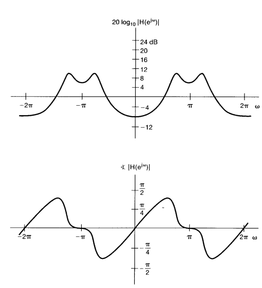
Bode Plots of CT System Responses

- The specific logarithmic amplitude scale used is in units $20\log_{10}$, referred to as ***decibels*** (dB)



- ***Bode plots*** are used in continuous-time in which a logarithmic frequency scale is commonly used

Log-Magnitude Plots of DT System Responses



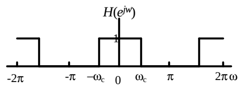
- In discrete-time, the magnitudes of the Fourier transform are often displayed in dB
- In discrete-time, the logarithmic frequency scale is not typically used because the range of frequencies is limited due to periodicity of the frequency response
- Notice that the relationship between amplitude and power gain in decibels is

$$20\log_{10}\left|H(e^{jw})\right| = 10\log_{10}\left|H(e^{jw})\right|^2$$

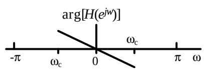
Amplitude gain — Power gain

Time-Domain Properties of Ideal Frequency-Selective Filters

$$H(e^{jw}) = \begin{cases} 1 & |w| \le w_c \\ 0 & w_c < |w| \le p \end{cases}$$



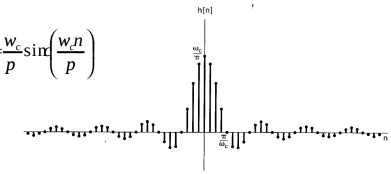
- Ideal lowpass filter has perfect selectivity
- In addition, the filter has zero phase characteristics, so they introduce no phase distortion
- An ideal filter with linear phase over the passband, introduces only a simple time shift when compared to the response of the ideal lowpass filter with zero phase characteristics



Time-Domain Properties of Ideal Frequency-Selective Filters

- The impulse response corresponding to the ideal lowpass filter in DT is the sinc function:

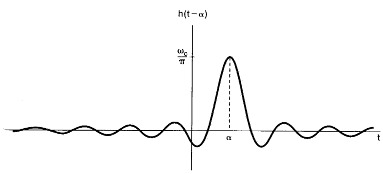
$$h[n] = \frac{\sin w_c n}{pn} = \frac{w_c}{p}\operatorname{sinc}\left(\frac{w_c n}{p}\right)$$



- Notice that the width of the the passband of $H(e^{jw})$ is proportional to w_c, while the width of the main lobe is proportional to $1/w_c$ in the impulse response $h[n]$

Time-Domain Properties of Ideal Frequency-Selective Filters

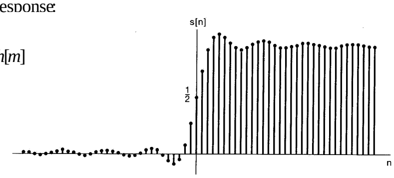
- Augmenting the ideal frequency response (either in CT ar DT case) with a linear phase characteristics simply delays the impulse response



Time-domain Properties of Ideal Frequency -Selective Filters

- The step response:

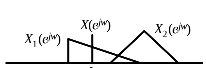
$$s[n] = \sum_{m=-\infty}^{n} h[m]$$



- The step response overshoots its longterm final values and oscillate
- The oscillatory behavior is referred to as ringing
- The rise time of the filter is inversely proportional to the bandwidth

Time -Domain and Frequency-Domain Aspects of Nonideal Filters

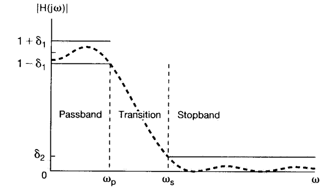
- In many filtering applications, the spectra of signals to be separated overlap slightly



- We may wish to trade off the fidelity with which the filter preserves one of the signals, e.g., $x_1(t)$, against the level to which frequency components of the second signal, $x_2(t)$, are attenuated
- A filter with a gradual transition from passband to stopband is generally preferable when filtering the superposition of signals with overlapping spectra

➡ ***Nonideal filters are of considerable practical importance***

Magnitude Specifications of a Nonideal Lowpass Filter

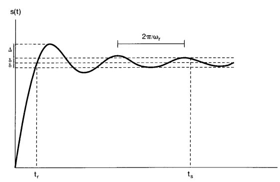


- Deviation in passband is d_1 and deviation in stopband is d_2
- Passband ripple specified by d_1 and stopband ripple specified by d_2
- Passband edge frequency is w_p and stopband edge frequency is w_s
- Transition band is the frequency range between w_p and w_s
- Specification of the phase characteristics is sometimes also important

The Time-Domain Behavior of Nonideal Filters

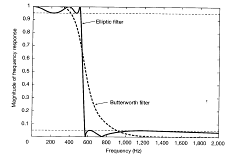
In time-domain, specifications are often imposed on the step response:

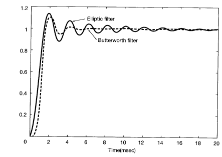
- Rise time t_r of the step response
- The oscillatory behavior, i.e. ringing, is often of importance
- Parameters characterizing the nature of ringing:



1) The overshoot Δ of the final value of the step response
2) The ringing frequency w_r
3) The settling time t_s, i.e. the time required for the step response to settle within a specified tolerance

Example 6.3: Fifth Order Filter Designed for w_p=500 Hz





- Rational frequency response and real-valued impulse response
- Butterworth filter: Wider transition band with less overshoot and ringing
- Elliptic filter: Narrower transition band with higher overshoot and longer settling time

First and Second Order Discrete-Time Systems

- Any system with a frequency response that is a ratio of polynomials in e^{jw}, i.e., any discrete-time LTI system described by a linear constant-coefficient difference equation, can be written as a product or sum of first and second order systems
- These basic systems are of considerable value in both implementing and analyzing more complex systems

First-Order Discrete-Time Systems

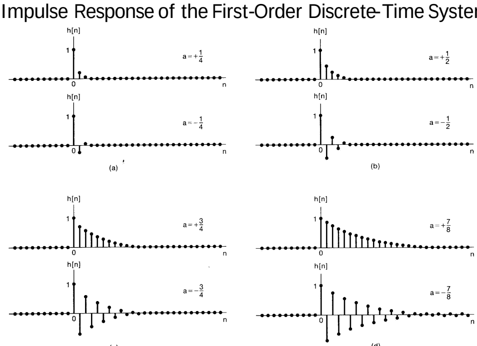
- First order causal LTI system

$$y[n]-ay[n-1]=x[n], \quad \text{with } |a|<1$$

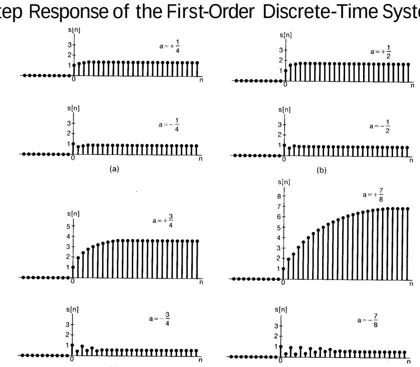
- The frequency response is: $H(e^{jw})=\dfrac{1}{1-ae^{-jw}}$
- The impulse response is: $h[n]=a^n u[n]$
- The step response is given as:

$$s[n]=h[n]*u[n]=\frac{1-a^{n+1}}{1-a}u[n]$$

Impulse Response of the First-Order Discrete-Time System



Step Response of the First-Order Discrete-Time System



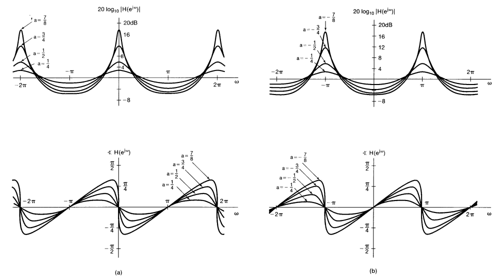
First-Order Discrete-Time Systems

- The magnitude of the frequency response of the first-order system is:

$$\left|H(e^{jw})\right| = \frac{1}{\left(1+a^2-2a\cos w\right)^{1/2}}$$

- The phase is: $\arg\left[H(e^{jw})\right] = -\tan^{-1}\frac{a\sin w}{1-a\cos w}$
- $|a|$ determines the rate at which the system responds
- For $a > 0$, the system attenuates high frequencies => lowpass filter
- For $a < 0$, the system attenuates low frequencies => highpass filter
- Responses are shown in Figure 6.28 (a) and (b)
- Parameter a plays similar role as the time-constant t in continuous-time systems

Frequency Response of the First-Order Discrete-Time System



Parameter a plays determines the behavior of the system

Second-Order Discrete-Time Systems

- Consider the second-order causal LTI system described by

$$y[n]-2r\cos q\, y[n-1]+r^2y[n-2]=x[n]$$

with $0 < r < 1$ and $0 \le q \le p$

- The frequency response for this system is:

$$H(e^{jw}) = \frac{1}{1-2r\cos q\; e^{-jw}+r^2e^{-j2w}}$$

- The denominator of $H(e^{jw})$ can be factored

$$H(e^{jw}) = \frac{1}{\left[1-(re^{jq})e^{-jw}\right]\left[1-(re^{-jq})e^{-jw}\right]}$$

Second-Order Discrete-Time Systems

- Partial fraction expansion gives (for q not equal to 0 or p)

$$H(e^{jw}) = \frac{A}{\left[1-(re^{jq})e^{-jw}\right]}+\frac{B}{\left[1-(re^{-jq})e^{-jw}\right]}$$

where $A = \frac{e^{jq}}{2j\sin q}$ and $B = \frac{e^{-jq}}{2j\sin q}$

- The impulse response of the system is now

$$h[n] = \left[A\left(re^{jq}\right)^n + B\left(re^{-jq}\right)^n\right]u[n] = r^n\frac{\sin[(n+1)q]}{\sin q}\,u[n]$$

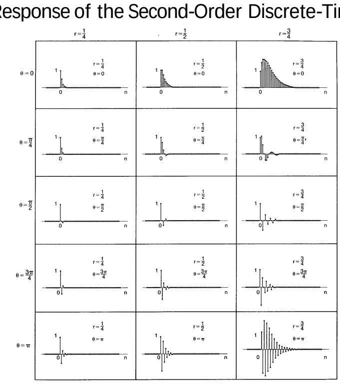
Second-Order Discrete-Time Systems

- When $q = 0$, $H(e^{jw}) = \frac{1}{\left[1-re^{-jw}\right]^2}$ and $h[n]=(n+1)r^nu[n]$
- When $q = p$, $H(e^{jw}) = \frac{1}{\left[1+re^{-jw}\right]^2}$ and $h[n]=(n+1)(-r)^nu[n]$

Impulse response:

- The rate of decay of $h[n]$ is controlled by r:
  The closer r is to 1, the slower is the decay in $h[n]$
- The frequency of oscillation is controlled by q
- Impulse responses with different parameter values are depicted in Fig. 6.29

Impulse Response of the Second-Order Discrete-Time System



Second-Order Discrete-Time Systems

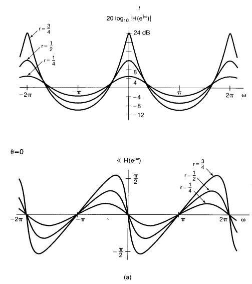
The step response:

- The effect of different values of r and q can also be seen by examining the step response
- Step responses with different parameter values are shown in Fig. 6.30
- For any value of q other than zero, the impulse response has a damped oscillatory behavior, and the step response exhibits ringing and overshoot

Magnitude and phase response:

- The band of frequencies determined by q is amplified
- The parameter r determines how sharply peaked the frequency response is
- Frequency responses are depicted in Fig. 6.31 (a)-(e)

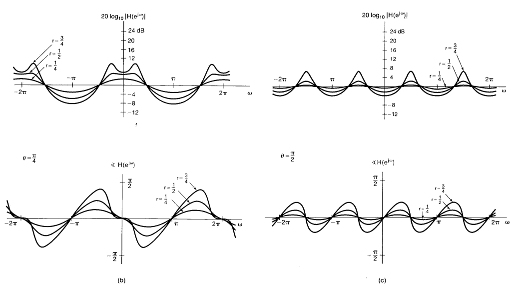
Frequency Response of the 2nd Order Discrete-Time System



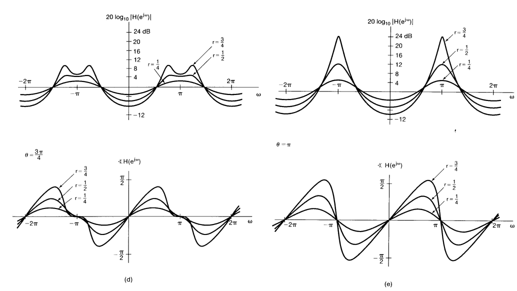
Magnitude and phase response:

- The band of frequencies determined by q is amplified
- The parameter r determines how sharply peaked the frequency response is
- The frequency with maximum gain is called the resonance frequency
- Frequency responses for different values of q are depicted in Fig. 6.31 (a)-(e)

Frequency Response of the 2nd Order Discrete-Time System



Frequency Response of the 2nd Order Discrete-Time System



Examples of Discrete-Time Nonrecursive Filters

- Moving average filter

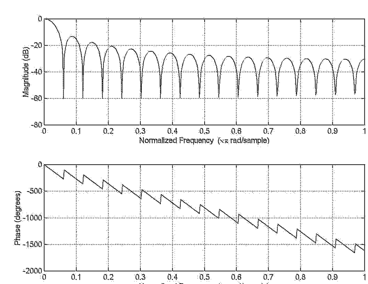
$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]$$

- The corresponding impulse response is a rectangular pulse
- The frequency response for this system is:

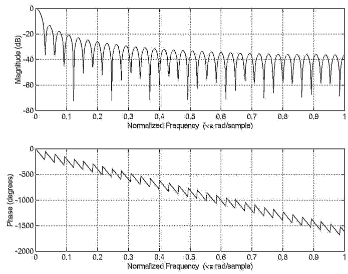
$$H(e^{jw}) = \frac{1}{N+M+1} e^{jw[(N-M)/2]} \frac{\sin[w(M+N+1)/2]}{\sin(w/2)}$$

- This corresponds to a lowpass behavior in frequency domain, i.e., it is the sinc function

Moving Average Filter (N=33)



Moving Average Filter (*N*=65)



Examples of Discrete-Time Nonrecursive Filters

- More general form of a discrete-time nonrecursive filter

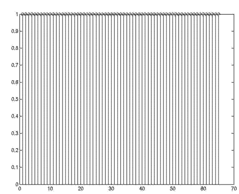
$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]$$

- The output of this filter can be thought as a weighted average of *N*+*M*+1 neighboring points
- In moving average filter all the weights are set to 1/(*N*+*M*+1)
- There is a variety of techniques available for choosing the coefficients b_k to meet certain specifications

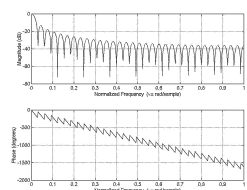
Window Function Method in FIR Filter Design

- In general, FIR filters can be designed using the ***window functions***, i.e., the infinite length impulse response resulting from the inverse Fourier transform of the frequency domain characteristics of an ideal lowpass filter is truncated by a window function with proper weights
- Rectangular window: $w_r[n] = \begin{cases} 1, & n = 0,1,\ldots,N-1 \\ 0, & \text{elsewhere} \end{cases}$
- Hanning window: $w_{Hanning}[n] = \frac{1}{2}\left[1 - \cos\frac{2pn}{N-1}\right]$
- Hamming window: $w_{Hamming}[n] = 0.54 - 0.46\cos\frac{2pn}{N-1}$
- Blackman window: $w_{Blackman}[n] = 0.42 - 0.5\cos\frac{2pn}{N-1} + 0.08\cos\frac{4pn}{N-1}$
- Kaiser window: Defined using the zero-order Bessel functions

Rectangular Window (*N*=65)

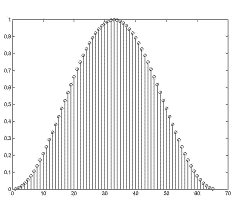


Impulse response

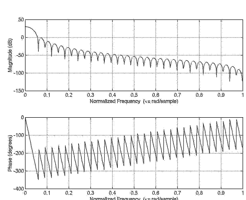


Frequency response

Hanning Window (*N*=65)

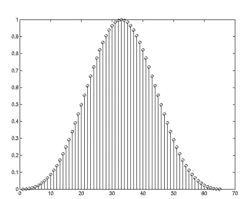


Impulse response

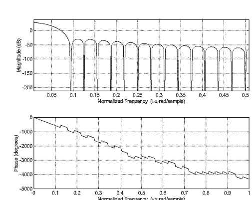


Frequency response

Blackman Window (*N*=65)

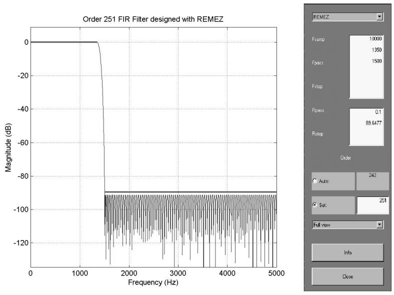


Impulse response

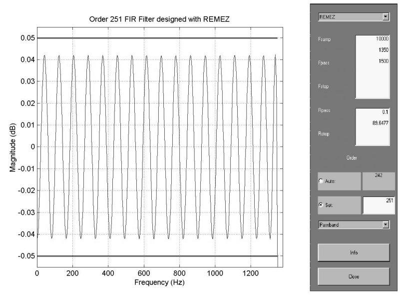


Frequency response

Example: High Order Lowpass FIR Filter



Example: High Order Lowpass FIR Filter / Passband



Moving Average Filter:

$$H(e^{jw}) = \frac{1}{N+M+1} e^{jw[(N-M)/2]} \frac{\sin[w(M+N+1)/2]}{\sin(w/2)}$$

- The phase is linear: $\arg[H(e^{jw})] = [(N-M)/2]w$
- From the symmetry properties of the Fourier transform of real signals we know that any nonrecursive filter with an impulse response that is real and even will have a frequency response $H(e^{jw})$ that is real and even and, consequently, has zero phase
- The filter is noncausal, since $h[n]$ has nonzero values for $n < 0$
- If a causal filter is required, then a simple change in the impulse response will result in linear phase response

Causal Moving Average Filter:

- Since $h[n]$ is the impulse response of an FIR filter, it is identically zero outside a range of values centered at the origin

$$h[n] = 0 \quad \text{for all} \quad |n| > N$$

- Now, define the nonrecursive LTI system obtained by delaying $h[n]$ with N steps (samples), i.e.

$$h_1[n] = h[n-N]$$

- The system defined by $h_1[n]$ is causal, i.e., $h_1[n]=0$ for $n < 0$
- The frequency response is obtained by the shifting property:

$$H_1(e^{jw}) = H(e^{jw})e^{-jwN}$$

- Since $H(e^{jw})$ has zero phase $H_1(e^{jw})$ does indeed have linear phase

Summary

- In this chapter, we have built on the foundation of Fourier analysis of signals and systems in order to examine in more detail the characteristics of LTI systems and the effects they have on signals
- Magnitude and phase charcteristics of signals and systems were discussed
- The impact of phase and phase distortion on signals and systems were considered
- Linear phase characteristics correspond to a constant delay at all frequencies
- Properties of ideal and nonideal frequency-selective filters were examined
- Time-frequency characteristics and tradeoffs of first and second order discrete-time recursive filters were investigated
- Various nonrecursive FIR filters based on windowing techniques were introduced