T-61.140 Signal Processing Systems

Exercise material for autumn 2003 - Solutions start from Page 16.

1. Basics of complex numbers (for example p. 71 / Oppenheim). Euler's formula

$$e^{j\omega} = \cos(\omega) + j\ \sin(\omega)$$

Express the following complex numbers in Cartesian form ($x + jy$):

a) $\frac{1}{2}e^{-j\pi}$
b) $e^{j5\pi/2}$

Express the following complex numbers in polar coordinates ($re^{j\theta}$, with $-\pi < \theta \leq \pi$):

c) -2
d) $1 + j$

Using complex conjugates $z = x + jy = r\,e^{j\theta}$, $z^* = x - jy = r\,e^{-j\theta}$ and module $|z| = r = (x^2 + y^2)^{1/2}$, show that

e) $zz^* = r^2$
f) $(z_1 + z_2)^* = z_1^* + z_2^*$

2. Even and odd functions. What is an even function ($\mathcal{E}ven$), odd ($\mathcal{O}dd$)? Sketch an example. Att: $\mathcal{E}ven\{x(t)\} = 1/2[x(t) + x(-t)]$ ja $\mathcal{O}dd\{x(t)\} = 1/2[x(t) - x(-t)]$. Calculate:

a) $H(\omega) = Even\{e^{j\omega}\} = 1/2[H(\omega) + H(-\omega)]$
b) $y(t) = Odd\{\sin(4\pi t)u(t)\}$

3. Sketch the following signals and sequences around origo ($t = 0$ or $n = 0$).

a) $x_1(t) = \cos(t - \pi/2)$
b) $x_2[n] = \sin(0.1\pi n)$
c) $x_3[n] = \sin(2\pi n)$
d) $x_4[n] = \delta[n-1] + \delta[n] + 2\delta[n+1]$
e) $x_5[n] = \delta[-1] + \delta[0] + 2\delta[1]$
f) $x_6[n] = u[n] - u[n-4]$

4. Which of the following continuous-time signals are periodic? Derive the basic period of periodic signals.

a) $x(t) = 3\cos(\frac{8\pi}{31}t)$
b) $x(t) = e^{j(\pi t - 1)}$
c) $x(t) = \cos(\frac{\pi}{8}t^2)$

5. Which of the following discrete-time sequences are periodic? Derive the basic period of periodic sequences.

a) $x[n] = 3\ \cos(\frac{8\pi}{31}n)$

b) $x[n] = \cos(\frac{n}{8} - \pi)$
c) $x[n] = 2\ \cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

6. Consider two systems S_1 and, S_2, whose input-outputrelations are:

$$S_1 : y[n] = x[n] + 2x[n-2]$$
$$S_2 : y[n] = x[n] - 3x[n-1] - 2x[n-2]$$

a) Express the output for the cascade S_1 ja S_2
b) Express the output for the parallel S_1 ja S_2

7. In Figure 1 there is a discrete-time system S, whose output is

$$y[n] = \mathcal{O}d\{x[n+1]\} = \frac{1}{2}(x[n+1] - x[-n-1])$$

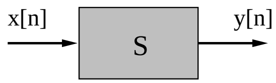

Figure 1: *Problem 7: System S.*

Is the system

a) memoryless ?	*b)* linear ?
c) time-invariant ?	*d)* stable ?
e) causal ?	

8. The output of a linear time-invariant system to an input $x_1(t)$ is $y_1(t)$ (see Figure 2). Calculate the output of the system with input $x_2(t)$.

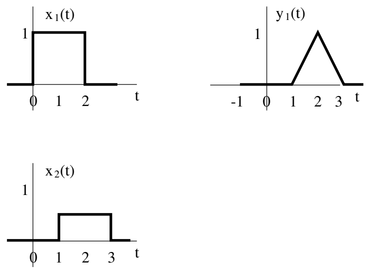

Figure 2: *Problem 8: The input and output of a linear time-invariant system.*

9. Calculate the convolution $h[n] * x[n]$ for

   a) $x[n]$ ja $h[n]$ are depicted in Figure 3. (LTI)

   b) $x[n] = \alpha^n u[n]$
   $h[n] = \beta^n u[n]$

   c) $x[n] = (-\frac{1}{2})^n u[n-4]$
   $h[n] = 4^n(2-n)$

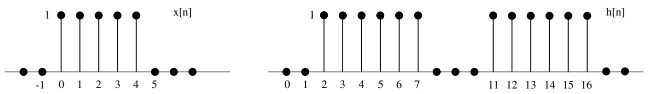

Figure 3: *Problem 9(a): The input and impulse response of the system.*

10. System properties. Examine, if the system below is

   a) linear and/or causal: $y[n] = a\,x[n] + b^2\,x[n-1] + ab\,x[n-2]$, where a and b are real coefficients

   b) stabiili and/or causal: $y[n] = x[n+2] + 0.5^n\ x[n+1]$

   c) linear and/or time-invariant: $y[n] = x^2[n] = (x[n])^2$

   d) time-invariant and/or stabiili: $y[n] = 2n\,x[n-1]$

   e) memoryless and/or invertible: $y[n] = x[1-n]$

   f) linear and/or invertible: $y[n] = x[n] + a$, where a is a real coefficient

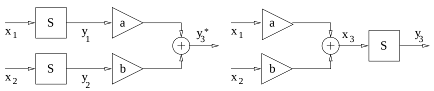

Figure 4: *Showing the linearity.*

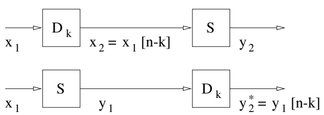

Figure 5: *Showing the time-invariance.*

11. Consider a system defined by the difference equation

$$y[n] = x[n] - x[n-1] \ .$$

   a) Sketch the block diagram of the system.

   b) Determine the impulse response $h[n]$ of the system.

   c) What is the response of the system to input sequence $x[n] = \left(\frac{1}{3}\right)^n u[n]$.

12. Consider a system defined by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] \ .$$

   a) Sketch the block diagram of the system.

   b) Determine the impulse response $h[n]$ of the system when $0 \le n \le 4$. What is the impulse response like with larger values of n?

   c) Solve the difference equation with the input $x[n] = \left(\frac{1}{3}\right)^n u[n]$.

13. Suppose we have a cascade connection of three linear and time-invariant (LTI) systems (Figure 6). It is known that the impulse response $h_2[n]$ equals

$$h_2[n] = u[n] - u[n-2]$$

and that the impulse response of the whole connection equals the one shown in Figure 7.

   a) What is the length of the non-zero portion of the impulse response $h_1[n]$? Determine the impulse response $h_1[n]$.

   b) What is the response of the system to input sequence $x[n] = \delta[n] - \delta[n-1]$?

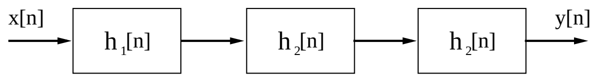

Figure 6: *Problem 13: A cascade of three LTI systems.*

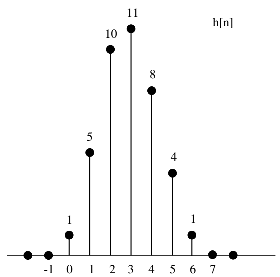

Figure 7: *Problem 13: The impulse response of the cascade system.*

14. Calculate the convolution

$$y[n] = x[n] * h[n]$$

where

$$x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$$
$$h[n] = \left(\frac{1}{4}\right)^n u[n+3]$$

a) directly by using the definition of convolution,

b) by using the distributive property of convolution ($(x_1 + x_2) * h = (x_1 * h) + (x_2 * h)$).

15. Suppose we have a LTI system S whose output $y[n]$ and input $x[n]$ can be characterized by the difference equation

$$2y[n] - y[n-1] + y[n-3] = x[n] - 5x[n-4] \ .$$

a) Verify, whether S can be represented as a cascade of two causal LTI systems S_1 ja S_2 defined as

$$S_1: \quad 2y_1[n] = x_1[n] - 5x_1[n-4]$$
$$S_2: \quad y_2[n] = \frac{1}{2}y_2[n-1] - \frac{1}{2}y_2[n-3] + x_2[n]$$

b) Sketch the block diagram of S_1.

c) Sketch the block diagram of S_2.

d) Sketch the block diagram of a cascade consisting of the systems S_1 and S_2 (in that order).

e) Sketch the block diagram of a cascade consisting of the systems S_2 and S_1 (in that order).

16. Examine systems S_1 and S_2 with complex input $e^{j\pi n/6}$. Does the information prove that the system S_1 or S_2 is not a LTI (Section 3.2)?

$$S_1 \ : \ e^{j\pi n/6} \rightarrow e^{j3\pi n/7}$$
$$S_2 \ : \ e^{j\pi n/6} \rightarrow 0.23\, e^{j\pi n/6}$$

17. Consider Fourier coefficients $a_0 = 0,\ a_1 = a_{-1} = 2,\ a_2 = a_{-2} = 0,\ a_3 = a_{-3} = -1,\ a_k = 0$ for other k. Form $x(t)$ with a synthesis equation, whose length of basic period is 4. (3.3)

18. Find Fourier coefficients

a) $x_1(t) = e^{-j\omega_0 t}$, (complex signal)

b) $x_2(t) = \cos(2\pi t) + \cos(3\pi t)$, (real signal)

19. Consider Fourier coefficients $a_0 = 1,\ a_1 = a_{-1} = 2,\ a_2 = a_{-2} = -1$ with basic period $N = 5$. Form $x[n]$. (3.6)

20. Find fundamental angular frequencies and Fourier coefficients

a) $x_1[n] = \cos(\pi n/3)$

b) $x_2[n] = \sin(\pi n/2) + \cos(\pi n/4)$

21. Periodic signal $x(t)$, whose fundamental period is 2 is defined:

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \end{cases}$$

a) Find Fourier coefficient a_0. What does it represent?

b) What is the Fourier series of the derivate $\frac{dx(t)}{dt}$

c) Use the result from b) and differentiation property of Fourier series (from the table: $\frac{dx(t)}{dt}$ ... $jk\omega_0 a_k$) and find the Fourier coefficients of $x(t)$.

22. Find Fourier coefficients for the following discrete-time signals.

a) $x[n]$ is like in Figure 8

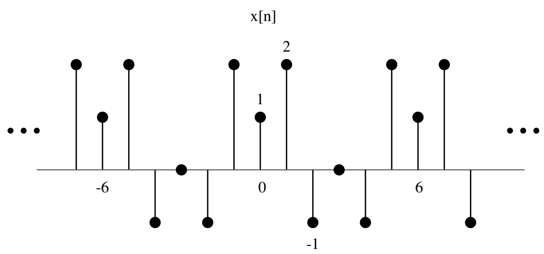

Figure 8: *Problem 22: discrete-time signal.*

b) $x[n] = \sin(2\pi n/3)\cos(\pi n/2)$

23. Suppose we know the following information about a signal $x(t)$:

i) $x(t)$ is real.

ii) $x(t)$ is periodic with period $T = 6$.

iii) $a_k = 0$, for $k = 0$ and $k > 2$.

iv) $x(t) = -x(t-3)$.

v) $\frac{1}{6}\int_{-3}^{3} |x(t)|^2 dt = \frac{1}{2}$.

vi) a_1 is real and positive.

Show that $x(t) = A\cos(Bt + C)$ and determine the constants A, B, and C.

24. Sketch the amplitude responses of the following filter types:

a) lowpass filter

b) highpass filter

c) bandstop filter

d) bandpass filter

25. Consider a continous periodic signal $x(t)$, defined as:

$$x(t) = \cos(2\pi t) + 0.3\cos(20\pi t)\ .$$

a) Sketch the signal $x(t)$ in time-domain.

b) Determine the fundamental angular frequency and Fourier coefficients a_k of the signal $x(t)$.

c) The signal $x(t)$ is filtered with an ideal lowpass filter having the cut-off frequency $\omega_c = 10\pi$.
Sketch the filtered signal.

d) The signal $x(t)$ is filtered with an ideal highpass filter having the cut-off frequency $\omega_c = 10\pi$.
Sketch the filtered signal.

26. Consider a mechanical system displayed in Figure 9. The differential equation relating velocity $v(t)$ and the input force $f(t)$ is given by

$$Bv(t) + K\int v(t)dt = f(t).$$

a) Assuming that the output is $f_s(t)$, the compressive force acting on the spring, write the differential equation relating $f_s(t)$ and $f(t)$. Obtain the frequency response of the system and argue that it approximates that of a lowpass filter.

b) Assuming that the output is $f_d(t)$, the compressive force acting on the dashpot, write the differential equation relating $f_d(t)$ and $f(t)$. Obtain the frequency response of the system and argue that it approximates that of a highpass filter.

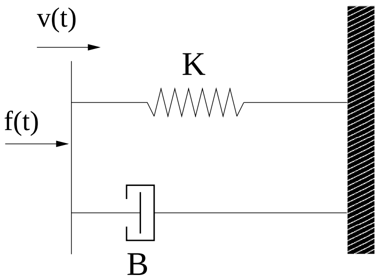

Figure 9: *Problem 26: The mechanical system.*

27. Fourier-transform, calculating integrals, and sinc-function. Examples 4.4 and 4.5 in the book.

a) Calculate Fourier-transform for a signal

$$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases}$$

b) Express the F-transform of a) with sinc-function, $\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

28. Calculate Fourier-transforms for the following signals and impulse responses. Use the result from 1) and tables 4.1 and 4.2. (time shifting, linearity, differentiation in time).

a) $x(t) = \begin{cases} 2, & 0 < t < 2 \\ 0, & \text{elsewhere} \end{cases}$

b) $x(t) = \begin{cases} 1, & 0 < t < 1 \\ 3, & 1 < t < 2 \\ 2, & 2 < t < 4 \\ 0, & \text{elsewhere} \end{cases}$

c) $h(t) = e^{-(t-2)}u(t-2)$

d) $x(t) = \begin{cases} 1 - |t|, & -1 < t < 1 \\ 0, & \text{elsewhere} \end{cases}$

29. Convolution property (p. 314)

$$y(t) = h(t) * x(t) \leftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

Calculate the convolution $h_1(t) * h_2(t)$ of impulse responses $h_1(t) = e^{-0.5t}u(t)$ and $h_2(t) = 2e^{-t}u(t)$ using the convolution property of F-transform (multiplication of transforms, inverse transform back to time domain).

30. Multiplication property (p. 322)

$$r(t) = s(t)p(t) \leftrightarrow R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

a) Let $X(j\omega)$ in Figure 10 be the spectrum of $x(t)$.

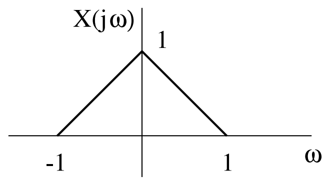

Figure 10: *Problem 30: Spectrum, Fourier-transform of the signal $x(t)$.*

Draw the spectrum $Y(j\omega)$ of $y(t) = x(t)p(t)$, when

i) $p(t) = cos(t/2),\ \omega = 0.5,\ T = 4\pi$

ii) $p(t) = cos(t),\ \omega = 1,\ T = 2\pi$

iii) $p(t) = cos(2t),\ \omega = 2,\ T = \pi$

iv) impulse train $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\pi),\ T = \pi,\ \omega = 2$, first find the F-coefficients of $p(t)$ (p. 299, example 4.8)

b) Calculate the F-transform for the signal $x(t) = \left(\frac{sin(\pi t)}{\pi t}\right)\left(\frac{sin(2\pi(t-1))}{\pi(t-1)}\right)$

31. Representing signals with the discrete-time Fourier transform. Calculate the Fourier transforms of the following signals:

   a) $x[n] = (\frac{1}{2})^{n-1}u[n-1]$
   b) $x[n] = \delta[n-1] + \delta[n+1]$

32. Properties of the discrete-time Fourier transform (see Table 5.1 p. 391 in the course book). Given that $X(e^{j\omega})$ is the Fourier transform of signal $x[n]$, express the Fourier transforms of the following signals in terms of $X(e^{j\omega})$.

   a) $x_1[n] = x[1-n] + x[-1-n]$
   b) $x_2[n] = \frac{1}{2}(x^*[-n] + x[n])$
   c) $x_3[n] = (n-1)^2 x[n]$

33. Suppose $X(e^{j\omega})$ is the Fourier transform of signal $x[n]$ shown in Figure 11.

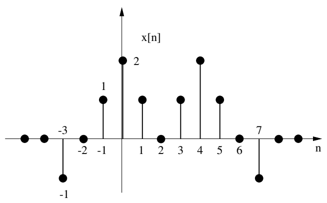

Figure 11: *Problem 33: A discrete signal $x[n]$.*

a) Calculate $X(e^{j0})$.

b) Find $\angle X(e^{j\omega})$.

c) Calculate $\int_{-\pi}^{+\pi} X(e^{j\omega})d\omega$.

d) Find $X(e^{j\pi})$.

e) Determine and sketch the signal, whose F-transfrom is $\Re e\{X(e^{j\omega})\}$.

f) Determine $\int_{-\pi}^{+\pi} |X(e^{j\omega})|^2 d\omega$ ja $\int_{-\pi}^{+\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega$.

Note! You don't have to determine the Fourier transform $X(e^{j\omega})$ itself in any of the problems above.

34. Determine which of the signals (a) - (f) in Figure 12 satisfy the following conditions

   1) $\Re e\{X(e^{j\omega})\} = 0$.
   2) $\Im m\{X(e^{j\omega})\} = 0$.
   3) There exists a real-valued α so that $e^{j\alpha\omega}X(e^{j\omega})$ is real.

   4) $\int_{-\pi}^{+\pi} X(e^{j\omega})d\omega = 0$.
   5) $X(e^{j\omega})$ is periodic.
   6) $X(e^{j0}) = 0$.

a) $x[n] = \left(\frac{1}{2}\right)^n u[n]$.

b) $x[n] = \left(\frac{1}{2}\right)^{|n|}$.

c) $x[n] = \delta[n-1] + \delta[n+2]$.

d) $x[n] = \sin\left(-n\frac{\pi}{2}\right)$.

e) $x[n] = \delta[n-1] + \delta[n+3]$.

f) $x[n] = \delta[n-1] - \delta[n+1]$.

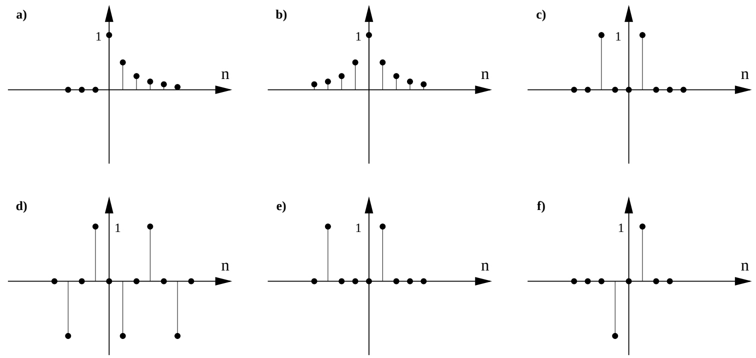

Figure 12: Problem 34: (a) - (f).

35. Suppose $X(e^{j\omega})$ and $G(e^{j\omega})$ are the Fourier transforms of signals $x[n]$ and $g[n]$, respectively. Additionally, we have the following equation is valid:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})G(e^{j(\omega-\theta)})d\theta = 1 + e^{-j\omega}$$

   a) Given that $x[n] = (-1)^n$, determine a discrete signal $g[n]$ whose Fourier transform $G(e^{j\omega})$ satisfies the above equation. Does other solutions for $g[n]$ exist?
   b) Consider the same problem for $x[n] = (\frac{1}{2})^n u[n]$.

36. Let there be a system with frequency response $H(e^{j\omega}) = 1 + e^{-j\omega}$. Frequency response can be decomposed to amplitude and phase responses $H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\ arg\{H(e^{j\omega})\}}$.

   a) Sketch $H(e^{j\omega})$ in complex plane, when ω gets values of $0..\pi$.
   b) Calculate the amplitude response $|H(e^{j\omega})|$ (absolute value of a complex number) and sketch it in range $0..\pi$. (Frequency in x-axis. In case of filters, their maximum value is scaled to unity.)
   c) Calculate phase response $arg\{H(e^{j\omega})\}$ (angle of a complex number) and sketch it in range $0..\pi$.

d) Decibels are often used. The transform is $20\,log_{10}|H(e^{j\omega})|$. Sketch amplitude response of b) in decibel-scale.

e) Group delay is the negation of the derivate of phase response $\tau(\omega) = -\frac{d}{d\omega} arg\{H(e^{j\omega})\}$. Calculate $\tau(\omega)$.

37. There is a frequency response $H(e^{j\omega})$ of a discrete-time sequence in range $0..\pi$ in Figure 13. Plotted with Matlab command `freqz`.

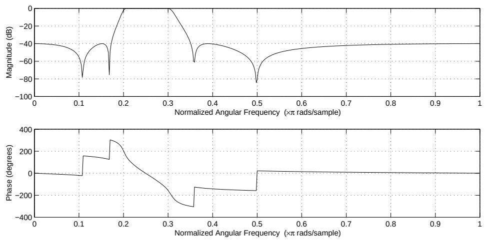

Figure 13: *Problem 37: $H(e^{j\omega})$: amplitude response and phase response*

a) Sketch the amplitude and phase response of $H(e^{j\omega})$ in range $-2\pi..2\pi$.

b) Why can you draw the result? (Hint: $H(e^{j\omega}) = \sum x[n]\,e^{-j\omega n}$, examine with $\omega_1 = \frac{\pi}{2}$ ja $\omega_2 = \frac{5\pi}{2}$.)

c) Why does the corresponding not work for continious-time Fourier-transforms (Bode diagram)?

38. Convolution in time domain corresponds multiplication of transforms in frequency domain (see Figure 14).

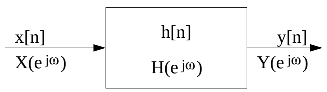

Figure 14: *Problem 38: $y[n] = h[n] * x[n] \leftrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$*

Let us know a LTI system S whose impulse function is $h[n] = \frac{1}{3}(-\delta[n] + 2\,\delta[n-1] - \delta[n-2])$.

a) Draw a block diagram of S (in time domain), where $x[n]$ comes from left into the system S (delay, sum, ...) and the output is $y[n]$.

b) Find $H(e^{j\omega})$ and sketch amplitude response $|H(e^{j\omega})|$. What is the filter like?

c) The response $y[n]$ can be gotten either with convolution $x[n]*h[n]$ in time domain or with inverse transform of multiplication of their Fourier transforms $F^{-1}\{F\{x[n]\}\ F\{h[n]\}\}$. Sketch the output $y[n]$, when the system is fed with sequences (time $t = 0$)

i) constant sequence, amplitude A, $\{A, A, A, ...\}$

ii) sequence $\{A, A, A, A, A, A, -A, -A, -A, -A, -A, -A, ...\}$

iii) periodic sequence $\{A, -A, A, -A, ...\}$

d) Can the system be realized? Is it causal and/or stable?

39. The frequency response of a continuous time, causal and stabile LTI-system is

$$H(j\omega) = \frac{1 - j\omega}{1 + j\omega}$$

a) Show that $|H(j\omega)| = A$, where A is constant. Calculate A.

b) What is the filter like?

c) Which of the following is true for the group delay $\tau(\omega) = -d(\arg H(j\omega))/d\omega$:

i) $\tau(\omega) = 0$, kun $\omega > 0$

ii) $\tau(\omega) > 0$, kun $\omega > 0$

iii) $\tau(\omega) < 0$, kun $\omega > 0$

40. Consider a continuous linear time invariant system, with frequency response

$$H(j\omega) = |H(j\omega)|e^{(j \arg H(j\omega))}$$

and the impulse response $h(t)$ is real. To input $x(t) = \cos(\omega_0 t + \phi_0)$ the output is $y(t) = Ax(t - t_0)$, where A is a non-negative real number and t_0 is a time delay.

a) Calculate A using $|H(j\omega_0)|$.

b) Calculate t_0 using the phase $\arg H(j\omega_0)$.

41. Sketch the phase-magnitude representation of F-transform of $x[n] = \cos(0.2\pi n) + 2\cos(0.05\pi n) + 0.1\,\epsilon[n]$, where $\epsilon[n]$ is gaussian white noise.

42. Linear and nonlinear phase. Examine the sequences $x_1 = cos(0.2\pi\, n)$ and $x_2 = 2\,cos(0.05\pi\, n)$ and the sum of these $x_3[n] = x_1[n] + x_2[n]$.

a) Draw the sequences x_1, x_2 and x_3.

b) Let there be a system S_1, whose group delay is constant $\tau_1(\omega) = 3$ and amplitude 1. Sketch the output for both sequences. Draw also x_3.

c) Let there be a system S_2, whose phase is nonlinear. Group delay is $\tau_2(0.05\pi) = 1$, $\tau_2(0.2\pi) = 5$ and amplitude constant 1. Sketch the output for both sequences. Draw also x_3. Compare results.

43. When designing highpass or bandpass filters, the standard method is to specify a lowpass filter having the desired characteristics and then convert it to a HP or BP filter. With this approach, we can use the lowpass design algorithms to design all filter types.

Let us consider a discrete-time lowpass filter with an impulse response $h_{lp}[n]$ and a frequency response $H_{lp}(e^{j\omega})$. Then we modulate the impulse response with the sequence $(-1)^n$ so that $h_{hp}[n] = (-1)^n h_{lp}[n]$.

a) Define $H_{hp}(e^{j\omega})$ using $H_{lp}(e^{j\omega})$. Show that $H_{hp}(e^{j\omega})$ is the frequency response of a highpass filter.

b) Show that modulating the impulse response of a discrete-time HP filter with the sequence $(-1)^n$ produces a LP filter.

44. The behavior of a linear time-invariant system is given by

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

a) Determine the frequency response of the system and sketch the corresponding Bode plot.

b) What is the group delay of the system?

c) Calculate the Fourier transform of the output of the system given the input $x(t) = e^{-t}u(t)$.

d) Calculate the output of the system given the input from c) using the partial fraction decomposition.

45. Suppose a non-ideal continous-time LP filter whose frequency response is $H_0(j\omega)$, impulse response $h_0(t)$, and step response $s_0(t)$. The cutoff frequency of the filter is $\omega_0 = 2\pi \times 10^2$ rad/s and the rise time of the step response (the amount of time in which the step response rises from 10% of its final value to 90% of it) is $\tau_r = 10^{-2}$ seconds. Let us implement a new filter with the frequency response

$$H_{lp}(j\omega) = H_0(ja\omega),$$

where a is the scaling factor.

a) Define a so that the cutoff frecuency of $H_{lp}(j\omega)$ is ω_c.

b) Present the impulse response $h_{lp}(t)$ of the new system with ω_c, ω_0, and $h_0(t)$.

c) Define the step response of the new system.

d) How do the cutoff frequency and the rise time of the new system relate to each other?

46. A causal and stable LTI system is defined with the following difference equation:

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n]$$

Determine

(a) the frecuency response $H(e^{j\omega})$

(b) the impulse response $h[n]$

47. Consider an ideal bandpass filter with the frequency response

$$H(j\omega) = \begin{cases} 1, & \omega_c \le |\omega| \le 3\omega_c \\ 0, & \text{otherwise} \end{cases}$$

(a) Given that $h(t)$ is the impulse response of the filter, define a function $g(t)$ so that

$$h(t) = (\frac{\sin \omega_c t}{\pi t})g(t)$$

(b) How does the impulse response change if the cutoff frequency ω_c is increased?

48. Explain the terms briefly: sampling process, impulse train, prefiltering, reconstruction of signal, zero order hold, aliasing

49. Show that a periodic impulse train $p(t)$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

can be expressed as a Fourier series

$$p(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j(2\pi/T)kt}$$

where $\Omega_T = 2\pi/T$ is sampling angular frequency. In other words, express $p(t)$ as Fourier series and find Fourier-coefficients for $p(t)$!

50. Sampling and aliasing

a) Find a value for angular frequency θ which satisfies

n	$\sin(\theta n)$
0	0
1	$-1/\sqrt{2}$
2	-1

What is θ in general?

b) Consider a continuous time periodic signal

$$x(t) = \begin{cases} \sin(2\pi f_1 t) + \sin(2\pi f_2 t) - \sin(2\pi f_3 t), & t \ge 0 \\ 0, & t < 0 \end{cases}$$

where f_1=100 Hz, f_2=300 Hz and f_3=700 Hz. The signal is sampled using frequency f_s, in other words, $T = 1/f_s$, $p(t) = f_s \sum_{k=-\infty}^{\infty} e^{j(2\pi f_s)kt}$. Thus, a discrete signal $x[n] = x_p(t) = x(nT)$ is obtained.
Sketch the magnitude of the Fourier spectrum of $x(n)$, the sampled signal, when f_s equals to

(i) 1500 Hz

(ii) 800 Hz

(iii) 400 Hz.

(Hint: a sampled sinusoid can be seen as a peak in the Fourier spectrum.)

c) With low sampling frequencies high frequencies of the signal aliased to low frequencies. When does this happen? How it is seen in the reconstructed signal?

51. Signal reconstruction from samples

a) Draw an arbitrary bandlimited $X(j\omega)$ whose biggest angular frequency is ω_M.

b) Sample the signal with sampling angular frequency $\omega_s > 2\,\omega_M$. Draw the spectrum of sampled $X_p(j\omega)$.

c) The sequence can be filtered. Reconstruct the signal using an ideal lowpass filter $H(j\omega)$, whose cut-off frequency ω_c is $\omega_M < \omega_c < 0.5\,\omega_s$. Draw the spectrum of reconstructed $X_r(j\omega) = X_p(j\omega)H(j\omega)$.

d) Express the equation of c) in time domain $x_r(t) = x_p(t) * h(t) = \sum_{k=-\infty}^{\infty} x_p(k)h(t - kT)$ What is the impulse response $h(t)$ of the ideal lowpass filter $H(j\omega)$ of c).

52. Why is it useful to sample a continuous signal and process it digitally? What problems occur when using too low sampling frequency. What about if sampling frequency is 1000 times bigger than highest frequency in signal?