T-61.3010 DSP Table of formulas, spring 2009

Disclaimer! Notations, e.g. ω or Ω, may vary from book to book, or from exam paper to other.

Basic math stuff

Even and odd functions:
$\mathcal{E}ven\{x(t)\} = 0.5 \cdot [x(t) + x(-t)]$
$\mathcal{O}dd\{x(t)\} = 0.5 \cdot [x(t) - x(-t)]$
Roots of second-order polynomial:
$ax^2 + bx + c = 0,\ x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$
Logarithms, decibels:
$\log\big((A \cdot B/C)^D\big) = D \cdot \big(\log A + \log B - \log C\big)$
$\log_a b = \log_c b / \log_c a$
decibels: $10\log_{10}(H^2/H_0^2) = 20\log_{10}(H/H_0)$
$10\log_{10}(0.5) = 20\log_{10}(\sqrt{0.5}) \approx -3.01$ dB
$20\log_{10}(0.1) = -20$ dB, $20\log_{10}(0.01) = -40$ dB
Complex numbers, radii, angles, unit circle:
$i \equiv j = \sqrt{-1} = -1/j$
$z = x + jy = r\,e^{j\theta}$
$r = \sqrt{x^2 + y^2}$
$\theta = \arctan(y/x) + n\pi,\ (n = 0,\ \text{if } x > 0, n = 1,\ \text{if } x < 0)$
$x = r\cos(\theta),\ y = r\sin(\theta)$
$e^{j\theta} = \cos(\theta) + j\sin(\theta)$ (*Euler's formula*)
$\cos(\theta) = (1/2)\cdot(e^{j\theta} + e^{-j\theta}),\ \sin(\theta) = (1/2j)\cdot(e^{j\theta} - e^{-j\theta})$
$z_1 \cdot z_2 = r_1 r_2 e^{j(\theta_1+\theta_2)},\ z_1/z_2 = (r_1/r_2)e^{j(\theta_1-\theta_2)}$
$|A \cdot B| = |A| \cdot |B|,\ \angle(A \cdot B) = \angle A + \angle B$
$z^n = r^n e^{jn\theta} = r^n(\cos\theta + j\sin\theta)^n = r^n(\cos n\theta + j\sin n\theta)$
$z_k = \sqrt[N]{z} = \sqrt[N]{r\,e^{j\theta}} = |\sqrt[N]{r}|\,e^{j(\theta+2\pi k)/N},\ k = 0, 1, \ldots, N-1$
Trigonometric functions:
$1° = \pi/180$ radians ≈ 0.01745 rad, 1 rad $= 180°/\pi \approx 57.30°$
$\mathrm{sinc}(\theta) = \sin(\pi\theta)/(\pi\theta)$
$\sin(\theta)/\theta \to 1,$ when $\theta \to 0$; $\mathrm{sinc}(\theta) \to 1,$ when $\theta \to 0$
$\cos^2(\theta) + \sin^2(\theta) = 1$
$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots + (-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \ldots$ (Taylor)
$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots + (-1)^n\frac{\theta^{2n}}{(2n)!} + \ldots$ (Taylor)

θ	0	$\pi/6$	$\pi/4$	$\pi/3$
$\sin(\theta)$	0	0.5	$\sqrt{2}/2$	$\sqrt{3}/2$
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	0.5
θ	$\pi/2$	$3\pi/4$	π	$-\pi/2$
$\sin(\theta)$	1	$\sqrt{2}/2$	0	-1
$\cos(\theta)$	0	$-\sqrt{2}/2$	-1	0

$\pi \approx 3.1416,\ \sqrt{3}/2 \approx 0.8660,\ \sqrt{2}/2 \approx 0.7071$
Geometric series:
$\sum_{n=0}^{+\infty} a^n = \dfrac{1}{1-a},\quad |a| < 1$
$\sum_{n=0}^{N} a^n = \dfrac{1-a^{N+1}}{1-a},\quad |a| < 1$

Continuous-time unit step and unit impulse fun.:

$$\mu(t) = \begin{cases} 1,\ t > 0 \\ 0,\ t < 0 \end{cases}$$

$\delta_\Delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mu_\Delta(t),\quad \delta(t) = \lim_{\Delta\to 0}\delta_\Delta(t)$ (Dirac's delta)
$\int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t = 1$
$\int_{-\infty}^{\infty}\delta(t-t_0)x(t)\,\mathrm{d}t = x(t_0)$
In DSP notation $2\pi\delta(t)$ is computed $2\pi\int\delta(t)\cdot 1\,\mathrm{d}t = 2\pi$, when $t = 0$, and $= 0$ elsewhere.

Discrete-time unit impulse and unit step functions:

$$\delta[n] = \begin{cases} 1,\ n = 0 \\ 0,\ n \neq 0 \end{cases} \qquad \mu[n] = \begin{cases} 1,\ n \geq 0 \\ 0,\ n < 0 \end{cases}$$

E.g. $x[n] = 2\delta[n+1] + \delta[n] - \delta[n-1] = \{2,\ \underline{1}, -1\}$, $x[-1] = 2,\ x[0] = 1,\ x[1] = -1.$

Periodic signals

$\exists T \in \mathbb{R} : x(t) = x(t+T), \forall t \in \mathbb{R}$
$\exists N \in \mathbb{Z} : x[n] = x[n+N], \forall n \in \mathbb{Z}$
Fundamental period T_0, N_0 is the smallest $T > 0$, $N > 0$.

Convolution

Convolution is commutative, associative and distributive.
$y(t) = h(t) \circledast x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$
$y[n] = h[n] \circledast x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$
$y_C[n] = h[n] \circledN x[n] = \sum_{k=0}^{N-1} h[k]x[<n-k>_N]$

Correlation:

$r_{xy}[l] = \sum_{n=-\infty}^{+\infty} x[n]y[n-l] = x[l] \circledast y[-l]$
$r_{xx}[l] = \sum_{n=-\infty}^{+\infty} x[n]x[n-l]$

Mean and variance of random signal:

$m_X = \mathrm{E}[X] = \int x p_X(x)dx$
$\sigma_X^2 = \int (x - m_X)^2 p_X(x)dx = \mathrm{E}[X^2] - m_X^2$

Frequencies, angular frequencies, periods:

Here f_s (also f_T later) is the sampling frequency.
Frequency f, $[f] = \mathrm{Hz} = 1/\mathrm{s}$.
Angular frequency $\Omega = 2\pi f = 2\pi/T$, $[\Omega] = \mathrm{rad/s}$ (analog).
Normalized angular frequency $\omega = 2\pi\Omega/\Omega_s = 2\pi f/f_s$, $[\omega] =$ rad/sample (digital).
Normalized frequency in Matlab $f_{MATLAB} = 2f/f_s$, $[f_{MATLAB}] = 1/\mathrm{sample}$.

Sampling of $x_a(t)$ by sampling frequency f_T

$x_p[n] = x_a(nT) = x_a(n/f_T)$
$X_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_T))$

Integral transforms. Properties

Here all integral transforms share some basic properties. Examples given with CTFT, $x[n] \leftrightarrow X(e^{j\omega})$, $x_1[n] \leftrightarrow X_1(e^{j\omega})$, and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are time-domain signals with corresponding transform-domain spectra. a and b are constants.
Linearity. All transforms are linear.
$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Time-shifting. There is a kernel term in transform, e.g.,
$x[n-k] \leftrightarrow e^{-jk\omega}X(e^{j\omega})$
Frequency-shifting. There is a kernel term in signal e.g.,
$e^{j\omega_k n}x[n] \leftrightarrow X(e^{j(\omega-\omega_k)})$
Conjugate symmetry $x^*[n] \leftrightarrow X^*(e^{-j\omega})$. If $x[n] \in \mathbb{R}$, then $X(e^{j\omega}) = X^*(e^{-j\omega})$, $|X(e^{j\omega})| = |X(e^{-j\omega})|$, $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$. If $x[n] \in \mathbb{R}$ and even, then $X(e^{j\omega}) \in \mathbb{R}$ and even. If $x[n] \in \mathbb{R}$ and odd, then $X(e^{j\omega})$ purely $\in \mathbb{C}$ and odd.
Time reversal. Transform variable is reversed, e.g.,
$x[-n] \leftrightarrow X(e^{-j\omega})$
Differentiation. In time and frequency domain, e.g.,
$x[n] - x[n-1] \leftrightarrow (1 - e^{-j\omega})X(e^{j\omega})$, $nx[n] \leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\omega}X(e^{j\omega})$
Duality. Convolution property: convolution in time domain corresponds multiplication in transform domain $x_1[n] \circledast x_2[n] \leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$ and multipication property, vice versa, $x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,\mathrm{d}\theta$
Parseval's relation. Energy in signal and spectral components: $\sum |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\,\mathrm{d}\omega$

Fourier series of continuous-time periodic signals:

$x(t) = \sum_{k=-\infty}^{\infty} a_k\,e^{jk\Omega_0 t}$ (synthesis)
$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\Omega_0 t}\,\mathrm{d}t$ (analysis)
$x(t - t_0) \leftrightarrow a_k e^{jk\Omega_0 t_0}$
$e^{jM\Omega_0 t}x(t) \leftrightarrow a_{k-M}$
$\int_T x_a(\tau)x_b(t-\tau)\,\mathrm{d}\tau \leftrightarrow T a_k b_k$
$x_a(t)x_b(t) \leftrightarrow \sum_l a_l b_{k-l}$
$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \leftrightarrow jk\Omega_0 a_k$

Continuous-time Fourier-transform (CTFT):

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\,e^{j\Omega t}\,\mathrm{d}\Omega$ (synthesis)
$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\Omega t}\,\mathrm{d}t$ (analysis)
$x(t - t_k) \leftrightarrow e^{j\Omega t_k}X(j\Omega)$
$e^{j\Omega_k t}x(t) \leftrightarrow X(j(\Omega - \Omega_k))$

$x_a(t) \circledast x_b(t) \leftrightarrow X_a(j\Omega)X_b(j\Omega)$
$x_a(t)x_b(t) \leftrightarrow \frac{1}{2\pi}X_a(j\Omega) \circledast X_b(j\Omega)$
$\frac{\mathrm{d}}{\mathrm{d}t}x(t) \leftrightarrow j\Omega X(j\Omega)$
$tx(t) \leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\Omega}X(j\Omega)$
$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega-\Omega_0)$
$\cos(\Omega_0 t) \leftrightarrow \pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$
$\sin(\Omega_0 t) \leftrightarrow j\pi[\delta(\Omega+\Omega_0)-\delta(\Omega-\Omega_0)]$
$x(t)=1 \leftrightarrow 2\pi\delta(\Omega)$
$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases} \leftrightarrow \frac{2\sin(\Omega T_1)}{\Omega}$
$\frac{\sin(Wt)}{\pi t} \leftrightarrow X(j\Omega) = \begin{cases} 1, |\Omega| < W \\ 0, |\Omega| > W \end{cases}$
$\delta(t) \leftrightarrow 1$
$\delta(t-t_k) \leftrightarrow e^{j\Omega t_k}$
$e^{-at}\mu(t) \leftrightarrow \frac{1}{a+j\Omega}$, where $\mathcal{R}eal\{a\} > 0$

Discrete-time Fourier-transform (DTFT):
$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\,\mathrm{d}\omega$ (synthesis)
$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$, periodic with 2π (analysis)
$x[n-k] \leftrightarrow e^{-jk\omega}X(e^{j\omega})$
$e^{j\omega_k n}x[n] \leftrightarrow X(e^{j(\omega-\omega_k)})$
$x_1[n] \circledast x_2[n] \leftrightarrow X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$
$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,\mathrm{d}\theta$
$nx[n] \leftrightarrow j\frac{\mathrm{d}}{\mathrm{d}\omega}X(e^{j\omega})$
$e^{j\omega_0 n} \leftrightarrow 2\pi\sum_l \delta(\omega-\omega_0-2\pi l)$
$\cos(\omega_0 n) \leftrightarrow \pi\sum_l[\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)]$
$\sin(\omega_0 n) \leftrightarrow j\pi\sum_l[\delta(\omega+\omega_0-2\pi l)-\delta(\omega-\omega_0-2\pi l)]$
$x[n]=1 \leftrightarrow 2\pi\sum_l \delta(\omega-2\pi l)$
$x[n] = \begin{cases} 1, |n| \le N_1 \\ 0, |n| > N_1 \end{cases} \leftrightarrow \frac{\sin(\omega(N_1+0.5))}{\sin(\omega/2)}$
$\frac{\sin(Wn)}{\pi n} = \frac{W}{\pi}\mathrm{sinc}(\frac{Wn}{\pi}) \leftrightarrow X(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}$
$\delta[n] \leftrightarrow 1$
$\delta[n-k] \leftrightarrow e^{-jk\omega}$
$a^n\mu[n] \leftrightarrow \frac{1}{1-ae^{-j\omega}}, \quad |a|<1$

N-point Discrete Fourier-transform (DFT):
Connection to DTFT: $X[k] = X(e^{j\omega})|_{\omega=2\pi k/N}$
$W_N = e^{-j2\pi/N}$
$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad 0 \le n \le N-1$ (synthesis)
$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad 0 \le k \le N-1$ (analysis)
$x[<n-n_0>_N] \leftrightarrow W_N^{-kn_0}X[k]$
$W_N^{-nk_0}x[n] \leftrightarrow X[<k-k_0>_N]$
$y_C[n] = h[n] \circledN x[n] \leftrightarrow H[k]\cdot X[k] = Y[k]$

Laplace transform:
Convergence with a certain ROC (region of convergence).
Connection to continuous-time Fourier-transform: $s = j\Omega$
$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,\mathrm{d}s$ (synthesis)
$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\,\mathrm{d}t$ (analysis)

z-transform:
Convergence with a certain ROC (region of convergence).
Connection to discrete-time Fourier-transform: $z = e^{j\omega}$
$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\mathrm{d}z, \quad C$ in ROC of $X(z)$ (synthesis)
$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ (analysis)
$x[n-k] \leftrightarrow z^{-k}X(z)$
$x_1[n] \circledast x_2[n] \leftrightarrow X_1(z)\cdot X_2(z)$
$\delta[n] \leftrightarrow 1$, ROC all z
$\delta[n-k] \leftrightarrow z^{-k}$, all z, except 0 ($k>0$) or ∞ ($k<0$)
$\mu[n] \leftrightarrow \frac{1}{1-z^{-1}}, \quad |z|>1$
$-\mu[-n-1] \leftrightarrow \frac{1}{1-z^{-1}}, \quad |z|<1$
$a^n\mu[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z|>|a|$
$na^n\mu[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z|>|a|$
$(n+1)a^n\mu[n] \leftrightarrow \frac{1}{(1-az^{-1})^2}, \quad |z|>|a|$
$r^n\cos(\omega_0 n)\mu[n] \leftrightarrow \frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}, \quad |z|>|r|$
$r^n\sin(\omega_0 n)\mu[n] \leftrightarrow \frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}, \quad |z|>|r|$

LTI filter analysis
Stability $\sum_n |h[n]| < \infty$; unit cirle belongs to ROC
Causality $h[n]=0, n<0$; ∞ belongs to ROC
Unit step response $s[n] = \sum_{k=-\infty}^{n} h[k]$
Causal transfer function of order $\max\{M,N\}$:
$H(z) = B(z)/A(z) = K\cdot\frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{n=0}^{N} a_n z^{-n}} = G\cdot\frac{\prod_{m=1}^{M}(1-d_m z^{-1})}{\prod_{n=1}^{N}(1-p_n z^{-1})}$
Zeros d_m: $B(z)=0$; *Poles* p_n: $A(z)=0$
Frequency, magnitude/amplitude, phase response, $z \leftarrow e^{j\omega}$
$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\angle H(e^{j\omega})}$
$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$
$H[k] = H(e^{j\omega})|_{\omega=2\pi k/N}$
Group delay $\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\angle H(e^{j\omega})$
Four types of linear-phase FIR filters, $h[n]=h[N-1-n]$ (even/odd symmetric), or $h[n] = -h[N-1-n]$ (e/o antis.).
Zeros symmetric w.r.t. unit circle: $r\, e^{\pm j\theta}$ and $(1/r)\, e^{\mp j\theta}$.
Important *transform pairs and properties*:
$a\,\delta[n-k] \leftrightarrow a\, e^{-jk\omega} \leftrightarrow a\, z^{-k}$
$a^n\mu[n] \leftrightarrow 1/[1-a\, e^{-j\omega}] \leftrightarrow 1/[1-a\, z^{-1}]$
$h[n] = \sum_i\big(k_i\cdot a_i^n\mu[n]\big) \leftrightarrow H(e^{j\omega}) = \ldots$
$\ldots\sum_i\big(k_i/[1-a_i\, e^{-j\omega}]\big) \leftrightarrow H(z) = \sum_i\big(k_i/[1-a_i\, z^{-1}]\big)$
$a\, x[n-k] \leftrightarrow a\, e^{-jk\omega}X(e^{j\omega}) \leftrightarrow a\, z^{-k}X(z)$
$y[n] = h[n] \circledast x[n] \leftrightarrow Y(z) = H(z)\cdot X(z)$
rectangular $\leftrightarrow$ sinc, sinc $\leftrightarrow$ rectangular

LTI filter design (synthesis)
Bilinear transform $H(z) = H(s)|_s$ and *prewarping*
$s = k\cdot(1-z^{-1})/(1+z^{-1}), \quad k=1$ or $k=2/T=2f_T$
$\Omega_{prewarp,c} = k\cdot\tan(\omega_c/2), \quad k=1$ or $k=2/T=2f_T$
Spectral transformations, $\hat{\omega}_c$ desired cut-off
LP-LP $z^{-1} = (\hat{z}^{-1}-\alpha)/(1-\alpha\hat{z}^{-1})$, where
$\alpha = \sin(0.5(\omega_c-\hat{\omega}_c))/\sin(0.5(\omega_c+\hat{\omega}_c))$
LP-HP $z^{-1} = -(\hat{z}^{-1}+\alpha)/(1+\alpha\hat{z}^{-1})$, where
$\alpha = -\cos(0.5(\omega_c+\hat{\omega}_c))/\cos(0.5(\omega_c-\hat{\omega}_c))$
Windowed Fourier series method
$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \ge \omega_c \end{cases} \leftrightarrow h[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi}\,\mathrm{sinc}(\frac{\omega_c n}{\pi})$
$h_{FIR}[n] = h_{ideal}[n]\cdot w[n]$
$H_{FIR}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{ideal}(e^{j\theta})W(e^{j(\omega-\theta)})\,\mathrm{d}\theta$
Fixed window functions, order $N=2M$, $-M \le n \le M$:
Rectangular $w[n]=1$
Hamming $w[n] = 0.54+0.46\cos((2\pi n)/(2M))$
Hann $w[n] = 0.5\cdot\big(1+\cos((2\pi n)/(2M))\big)$
Blackman $w[n] = 0.42+0.5\cos(\frac{2\pi n}{2M})+0.08\cos(\frac{4\pi n}{2M})$
Bartlett $w[n] = 1-(|n|/M)$

Implementation
Radix-2 DIT FFT butterfly equations
$\begin{cases} \Psi_{r+1}[\alpha] & = \Psi_r[\alpha] + W_{N_r}^l\,\Psi_r[\beta] \\ \Psi_{r+1}[\beta] & = \Psi_r[\alpha] - W_{N_r}^l\,\Psi_r[\beta] \end{cases}$
with Ψ_1 input, and Ψ_R, $R = \log_2 N + 1$ output level; $W_{N_r} = e^{-j2\pi/N_r}$, $N_r = 2^r$, $l \in [0,\, 2^{r-1}-1]$.

Multirate systems
Upsampling (interpolation) with factor L, $\boxed{\uparrow L}$
$x_u[n] = \begin{cases} x[n/L], n = 0, \pm L, \pm 2L, \ldots \\ x_u[n] = 0, \text{ otherwise} \end{cases}$
$X_u(z) = X(z^L)$, $X_u(e^{j\omega}) = X(e^{j\omega L})$
Downsampling (decimation) with factor M, $\boxed{\downarrow M}$
$x_d[n] = x[nM]$
$X_d(z) = (1/M)\sum_{k=0}^{M-1} X(z^{1/M}W_M^{-k})$,
$X_d(e^{j\omega}) = (1/M)\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$