13 Multirate Digital Signal Processing Fundamentals

Introduction

- **Single-rate systems:**

  Sampling rates at the input and at the output and all internal nodes are the same
- **Multirate systems:**

  DSP systems with unequal sampling rates at various parts of the system

Applications

- There are applications where the signal of a given sampling rate needs to be converted into an equivalent signal with a different sampling rate
- Sampling rates in some applications:
  - Digital audio applications:
    - 32 kHz in broadcasting,
    - 44,1 kHz in digital CD,
    - 48 kHz in digital audio tape (DAT)
  - Composite video signals:
    - NTSC: 14,3181818 MHz
    - PAL: 17,734475 MHz
  - Digital component video:
    - Luminance 13,5 MHz
    - Color difference 6,75 MHz

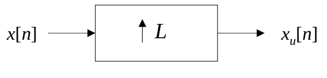
Multirate DSP Systems

- ***Down-sampler*** and ***up-sampler*** are the basic sampling rate alteration devices to achieve different sampling rates in multirate DSP systems
- Cascade connections of the basic sampling rate alteration devices and digital (lowpass) filters are used
- Down-sampling corresponds to ***decimation***
- Up-sampling corresponds to ***interpolation***

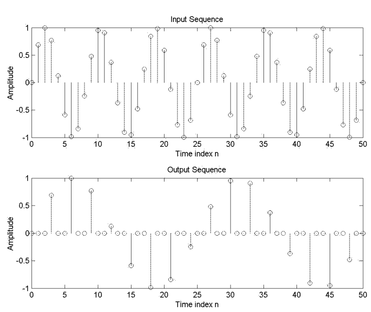
Up-Sampling in Time-Domain

- In up-sampling by an integer factor $L>1$, $L-1$ equidistant zero-valued samples are inserted between two consecutive samples of the input sequence $x[n]$:

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, ..., \\ 0, & \text{otherwise} \end{cases}$$

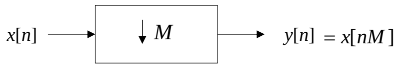


Up-Sampling Process (L=3)



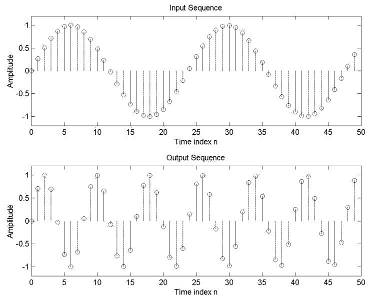
Down-Sampling in Time-Domain

- Down-sampling operation by an integer factor M>1 on sequence $x[n]$ consists of keeping every Mth sample of $x[n]$ and removing M-1 in-between samples, generating an output sequence $y[n]$:

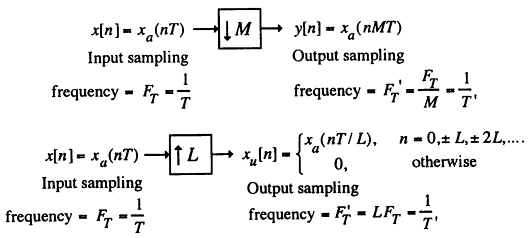


- Down-sampling results in a sequence $y[n]$ whose sampling rate is $(1/M)$th of that of $x[n]$

Down-Sampling Process (M=3)

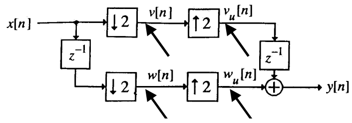


Building Blocks of Multirate Systems



The sampling rates are explicitly shown

Simple Multirate System



n:	0	1	2	3	4	5	6	7	8
$x[n]$:	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$	$x[8]$
$v[n]$:	$x[0]$	$x[2]$	$x[4]$	$x[6]$	$x[8]$	$x[10]$	$x[12]$	$x[14]$	$x[16]$
$w[n]$:	$x[-1]$	$x[1]$	$x[3]$	$x[5]$	$x[7]$	$x[9]$	$x[11]$	$x[13]$	$x[15]$
$v_u[n]$:	$x[0]$	0	$x[2]$	0	$x[4]$	0	$x[6]$	0	$x[8]$
$w_u[n]$:	$x[-1]$	0	$x[1]$	0	$x[3]$	0	$x[5]$	0	$x[7]$
$v_u[n-1]$:	0	$x[0]$	0	$x[2]$	0	$x[4]$	0	$x[6]$	0
$y[n]$:	$x[-1]$	$x[0]$	$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$	$x[6]$	$x[7]$

$$y[n] = v_u[n-1] + w_u[n] = x[n-1]$$

Frequency-Domain Characterization of Up-Sampling

- Consider a factor-of-2 up-sampler

$$x_u[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, ..., \\ 0, & \text{otherwise} \end{cases}$$

- In terms of the z-transform the input-output relation is

$$X_u(z) = \sum_{n=-\infty}^{\infty} x_u[n] z^{-n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] z^{-2m} = X(z^2)$$

Frequency-Domain Characterization

- In a similar manner, for the factor-of-L up-sampler

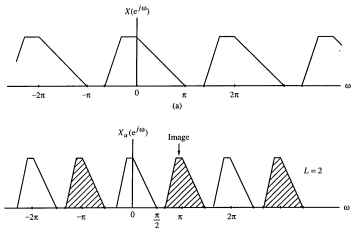
$$X_u(z) = X(z^L)$$

- Let us examine the above relation on the unit circle; For $z=e^{j\omega}$ the above equation becomes

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

- The factor-of-L up-sampling results in L-fold repetition of the DTFT

Effects of Up-Sampling in the Frequency-Domain



Input spectrum

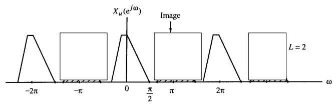
Output spectrum for L=2

- An additional "image" of the input spectrum appears

⇒ ***Imaging***

Effects of Up-Sampling in the Frequency-Domain

- In the case of factor-of-L sampling rate expansion there will be L-1 additional images of the input spectrum in the baseband
- Thus, the spectrum $X(e^{j\omega})$ of a bandlimited lowpass signal does not look like a low-frequency spectrum after up-sampling



- Lowpass filtering of the up-sampled signal $x_u[n]$ removes the images and "fills in" the zero-valued samples in $x_u[n]$ with interpolated sample values

Frequency-Domain Characterization of Down-Sampling

- Applying z-transform to input-output relation of down-sampling

$$Y(z) = \sum_{n=-\infty}^{\infty} x[nM]z^{-n}$$

- In order to express the right-hand side in terms of $X(z)$, let us define an intermediate sequence $x_{int}[n]$

$$x_{\text{int}}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M, \ldots, \\ 0, & \text{otherwise} \end{cases}$$

Frequency-Domain Characterization of Down-Sampling

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn]z^{-n} = \sum_{n=-\infty}^{\infty} x_{\text{int}}[Mn]z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_{\text{int}}[Mk]z^{-k/M} = X_{\text{int}}(z^{1/M})$$

- This can be written in the form (Mitra (13.9-13.11))

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^{-k}), \quad where\, W_M = e^{-j2\pi/M}$$

Effects of Down-Sampling in the Frequency-Domain

The spectrum of $y[n]$ is: $Y(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega/2}) + X(-e^{j\omega/2})\}$

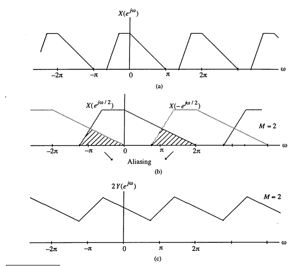


Illustration of the aliasing effect in the frequency domain caused by downsampling.

Spectrum of $x[n]$

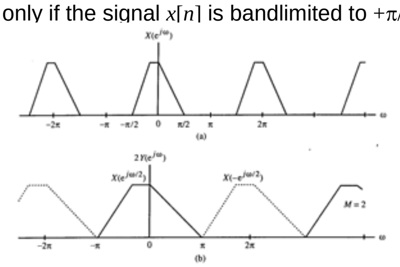
The plot of $X(e^{j\omega/2})$

The plot of $X(-e^{j\omega/2})$

Aliased spectrum due to down-sampling

Effects of Down-Sampling in the Frequency-Domain

- Aliasing due to a factor-of-M down-sampling is absent if and only if the signal $x[n]$ is bandlimited to $\pm\pi/M$

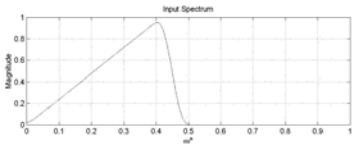


Effect of decimation in the frequency domain illustrating absence of aliasing.

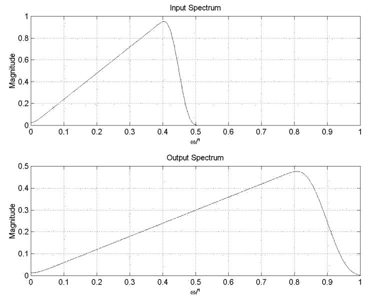
Frequency-Domain Characterization of Down-Sampling

- In down-sampling the spectrum $Y(e^{j\omega})$ is a sum of M uniformly shifted and stretched versions $X(e^{j\omega})$ of and then scaled by $1/M$

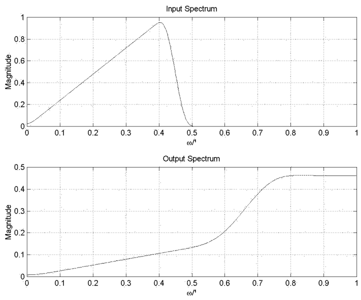
$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j\frac{(\omega-2\pi k)}{M}})$$



Example: Down-Sampling Factor of M=2



Example: Down-Sampling Factor of M=3

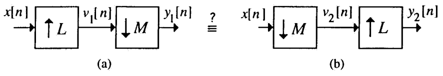


Cascade Equivalences

- Complex multirate systems consist of the basic sampling rate alteration devices and LTI digital filters
- In many applications, these devices appear in cascade connection
- Computationally efficient structures are often obtained by interchanging the order of cascaded blocks
- Specific cascade connections and their equivalences are investigated
- Basic sampling rate alteration devices can be used for integer factors only

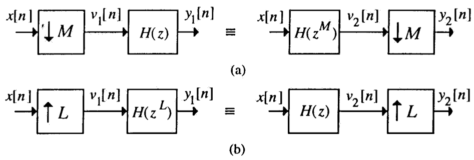
Cascade Equivalences

- Fractional change in the sampling rate can be implemented with a cascade of down-sampler and up-sampler
- Under which conditions the order of the cascaded blocks can be *interchanged* with no change in the input-output relation ?



Two different cascade arrangements of a down-sampler and an up-sampler.

Cascade Equivalences



Cascade equivalences: (a) Equivalence #1, and (b) Equivalence #2.

- The structure of the sampling rate alteration devices can be changed in order to achieve more efficient implementations

Filters in Multirate Systems

- The sampling rate of a critically sampled discrete-time system cannot be reduced without aliasing
- Before down-sampling the bandwidth of critically sampled signal must be reduced by *lowpass filtering*
- Similarly, the zero-valued samples introduced by an up-sampler must be interpolated to more appropriate values for an effective sampling rate increase
- Interpolation can be achieved by *lowpass filtering*

⇨ ***Lowpass filters are needed in multirate systems***

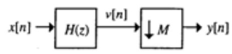
Filters in Multirate Systems

- In up-sampling, a lowpass filter is used after the up-sampler to remove the unwanted images of the spectrum

⇨ ***Interpolation filter***

- In down-sampling, a lowpass filter is used before the down-sampler to reduce the frequency band in order to avoid aliasing

⇨ ***Decimation filter***



Interpolation Filter Specifications

- The Fourier transforms of the $x_a(t)$ and $x[n]$ are related as

$$X(e^{j\omega}) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T_0}\right)$$

- Since the sampling is done at the Nyquist rate, there is no overlap in the spectra of $X_a(j\omega/T_0)$
- If the sampling rate is much higher, $T = T_0/L$, yielding , its Fourier transform $Y(e^{j\omega})$ is related to $X_a(j\Omega)$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T}\right) = \frac{L}{T_0}\sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T_0/L}\right)$$

Interpolation Filter

- If $x[n]$ is passed through a factor-of-L up-sampler generating $x_u[n]$ the relation between $X_u(e^{j\omega})$ and $X(e^{j\omega})$ is given by

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

- If $x_u[n]$ is passed through an ideal lowpass filter with a cutoff frequency at π/L and gain L, the output of the filter is precisely $y[n]$
- In practice, a transition band is provided to ensure the realizability and stability of the lowpass interpolation filter $H(z)$
- The desired lowpass filter should have a stopband edge at $\omega_s = \pi/L$ and passband edge ω_p close to ω_s to reduce the distortion of the spectrum of the signal $x[n]$

Filter Specifications

- If ω_c denoted the highest frequency that need to be preserved in the signal to be interpolated, the passband edge ω_p of the lowpass filter should be at $\omega_p = \omega_c/L$
- Summarizing:

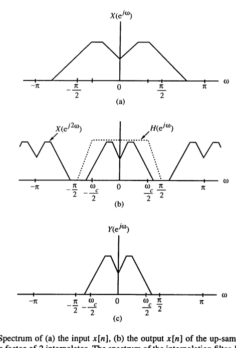
$$|H(e^{j\omega})| = \begin{cases} L, & |\omega| \le \omega_c/L \\ 0, & \pi/L \le |\omega| \le \pi \end{cases}$$

- In a similar manner, the specifications for the lowpass decimation filter can be developed

$$|H(e^{j\omega})| = \begin{cases} 1, & |\omega| \le \omega_c/M \\ 0, & \pi/M \le |\omega| \le \pi \end{cases}$$

where ω_c is the highest frequency needed to be preserved in the decimated signal

Interpolation in Frequency Domain (L=2)



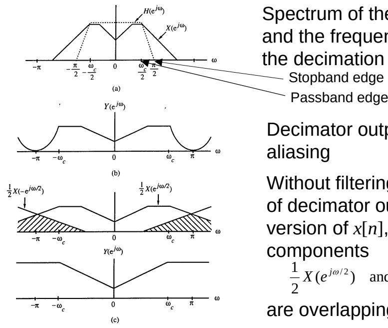
Spectrum of the input signal $x[n]$

Spectrum of the interpolator output $v[n]$: $V(e^{j\omega}) = X(e^{j2\omega})$

The spectrum of interpolator output is obtained by filtering $v[n]$, i.e., removing the images from the spectrum

Figure 10.17 Spectrum of (a) the input $x[n]$, (b) the output $x[n]$ of the up-sampler, and (c) the output $y[n]$ of the factor-of-2 interpolator. The spectrum of the interpolation filter $H(z)$ is shown in (b) with dotted lines.

Decimation in Frequency Domain (M=2)



Spectrum of the input signal $x[n]$ and the frequency response of the decimation filter

Decimator output without aliasing

Without filtering the spectrum of decimator output is aliased version of $x[n]$, i.e., the two components

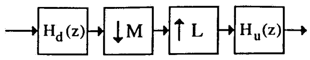
$$\frac{1}{2}X(e^{j\omega/2}) \quad \text{and} \quad \frac{1}{2}X(-e^{j\omega/2})$$

are overlapping

Figure 10.16 Spectrum of (a) the input $x[n]$, (b) the output $y[n]$ of the factor-of-2 decimator with $x[n]$ filtered, and (c) the output $y[n]$ of the factor-of-2 decimator with no filtering of $x[n]$ showing the effect of aliasing. The spectrum of the decimation filter $H(z)$ is shown in (a) with dotted lines.

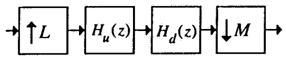
Filters for Fractional Sampling rate Alteration

- A fractional change in the sampling rate is achieved by cascading a factor-of-M with a factor-of-L interpolator
- The cascade is equivalent to
  - A decimator with a decimation factor of M/L

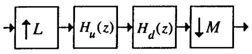


or

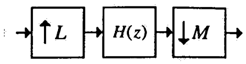
  - An interpolator with an interpolation factor of L/M



Filters for Fractional Sampling rate Alteration



- The two filters are in cascade are both lowpass filters
- Thus, only the one with lower stopband frequency is needed

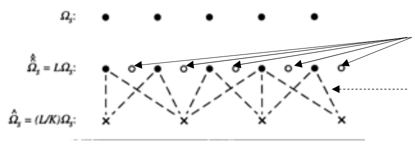


- The stopband cutoff frequency of the lowpass filter $H(z)$ is

$$\omega_s = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$$

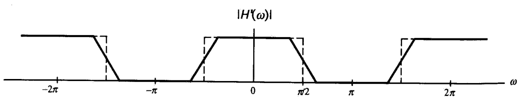
Computational Requirements

- The interpolation and decimation filters can be designed either as FIR or IIR filters
- FIR filters are usually preferred because interpolation introduces several zero-valued samples which need not be “multiplied” by FIR filter coefficients
- Example: Sampling rate change by a factor 2/3



Halfband Filters

- For the sampling rate change by a factor-of-2 (L=2 or M=2), the appropriate lowpass filter is a ***halfband*** filter



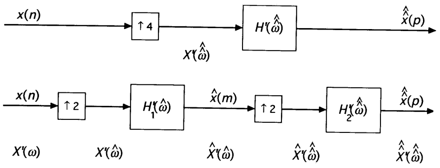
- The periodic frequency response of the halfband filter is an approximation to a square wave in the frequency domain
- The corresponding (noncausal) Fourier series $h[n]$ equals zero for n even (except n=0) and falls off approximately as $1/n$ for n odd

Computational Efficiency of the Halfband Filters

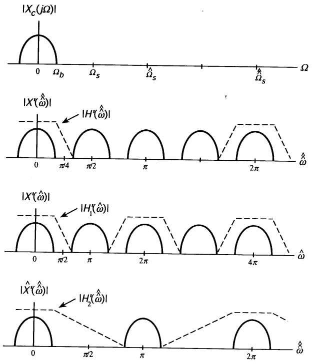
- The corresponding (noncausal) Fourier series $h[n]$ equals zero for n even (except n=0) and falls off approximately as $1/n$ for n odd
- If the filter is designed to preserve the odd symmetry of the passband and the stopband, the coefficients $h[n]$ for n even will remain zero
- The multiplication rate is reduced by a factor of almost two
- The odd symmetry of $H'(e^{j\omega})$ about $\pi/2$ can be preserved by the FIR design techniques

Narrowband Filters

- For very narrowband filters (L>>1 or M<<1) efficient sampling rate changes can be implemented in several stages
- For example, interpolation with a factor-of-4 can be be done in two stages with a factor-of-2 interpolators



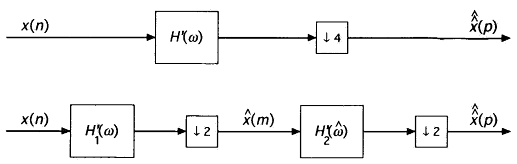
Two-Stage Interpolation



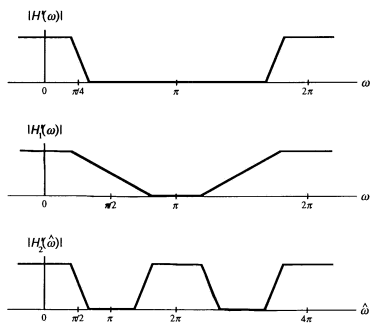
- Interpolation with a factor-of-4
- Interpolation filter designed with a sampling frequency $4\Omega_s$
- Two stages:
- $H_1(e^{j\omega})$ designed with a sampling frequency $2\Omega_s$
- $H_2(e^{j\omega})$ designed with a sampling frequency $4\Omega_s$

Narrowband Filters

- Decimation with a factor-of-4 can be be done in two stages with a factor-of-2 interpolators



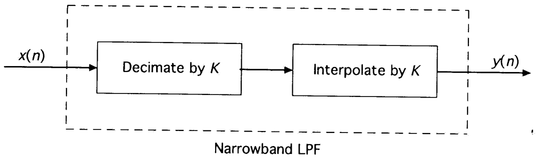
Two-Stage Decimation



- M=4: decimation filter designed with a sampling frequency $4\Omega_s$
- Two stages for M=4:
- $H_1(e^{j\omega})$ reduces the bandwidth to $\Omega_s/2$
- $H_2(e^{j\omega})$ operating at the sampling frequency $\Omega_s/2$ reduces the bandwith further to $\Omega_s/4$

Narrowband Lowpass Filters

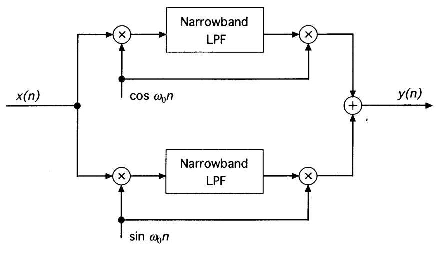
- Narrowband lowpass filters can be implemented by multistage decimation and interpolation



- If K is highly composite number the decimation and interpolation can be performed in many stages

Narrowband Bandpass Filters

- The center frequency ω_0 can be translated to dc by modulating the input signal by $e^{-j\omega_0}$ and by translating the lowpass filtered signal back to ω_0



The Polyphase Decomposition

- A single stage decimator or interpolator employing FIR filters can be computationally efficient since the necessary multiplications are required to compute the output samples can be carried out only when needed
- The computational requirements can be further decreased by using multistage designs
- Additional reduction in the computational complexity is possible by realizing the FIR filters using the polyphase decomposition

The Decomposition

- Consider an arbitrary sequence $\{x[n]\}$ with a z-transform $X(z)$: $$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$
- $X(z)$ can be rewritten as $$X(z)=\sum_{k=0}^{M-1}z^{-k}X_k\left(z^M\right)$$

$$X_k(z)=\sum_{k=0}^{M-1}x_k[n]z^{-n}=\sum_{n=-\infty}^{\infty}x[Mn+k]z^{-n},\quad k=0,1,2,...,M-1$$

- The subsequences $\{x_k[n]\}$ are called the polyphase components of the parent sequence $\{x[n]\}$
- The functions $X_k(z)$ given by the z-transforms of $\{x_k[n]\}$, are called the polyphase components of $X(z)$

The Decomposition

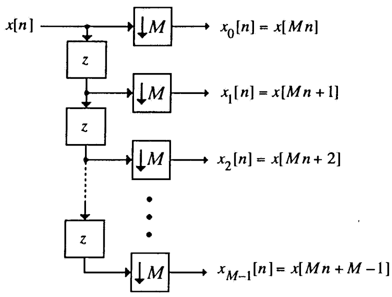
- The relation between the subsequences $\{x_k[n]\}$ and the original sequence $x[n]$ is given by

$$x_k[n]=x[Mn+k],\quad k=0,1,2,...,M-1$$

- $X(z)$ can be written in matrix form as

$$X(z)=\begin{bmatrix}1 & z^{-1} & ... & z^{-(M-1)}\end{bmatrix}\begin{bmatrix}X_0(z^M)\\ X_1(z^M)\\ :\\ X_{M-1}(z^M)\end{bmatrix}$$

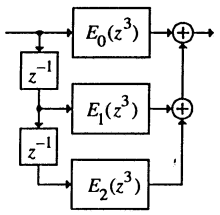
Structural Interpretation of the Polyphase Decomposition



Polyphase Realization of FIR Filters

General polyphase decomposition: $H(z)=\sum_{m=0}^{L-1}z^{-m}E_m(z^L)$

Example: $H(z)=E_0(z^3)+z^{-1}E_1(z^3)+z^{-2}E_2(z^3)$

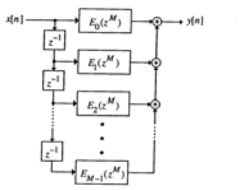


$E_0(z)=h[0]+h[3]z^{-1}+h[6]z^{-2}$
$E_1(z)=h[1]+h[4]z^{-1}+h[7]z^{-2}$
$E_2(z)=h[2]+h[5]z^{-1}+h[8]z^{-2}$

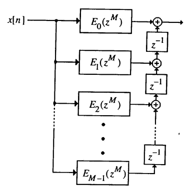
Parallel realization of an FIR filter using the polyphase decomposition

FIR Filter Structures Based on the Polyphase Decomposition

- Consider an M-branch polyphase decomposition of $H(z)$ given by $$H(z)=\sum_{k=0}^{M-1}z^{-k}E_k(z^M)$$



Type I polyphase realization



Transpose structure

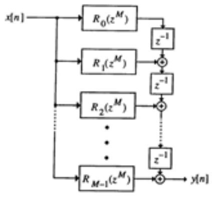
FIR Filter Structures Based on the Polyphase Decomposition

- Alternative representation of the transpose structure of the Type I polyphase decomposition is obtained using the notation

$$R_l(z^M) = E_{M-1-l}(z^M)$$
$$l = 0,\ 1,\ 2,...,\ M-1$$

- The corresponding polyphase decomposition is given as

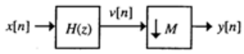
$$H(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_l(z^M)$$



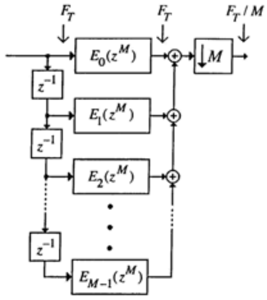
Type II polyphase decomposition

Computationally Efficient Interpolator and Decimator Structures

- Consider the use of the polyphase decomposition in the realization of the decimation filter

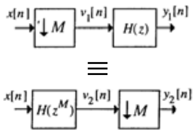


- If the lowpass filter is realized as a Type I polyphase decomposition the decimator structure is as shown on the right:

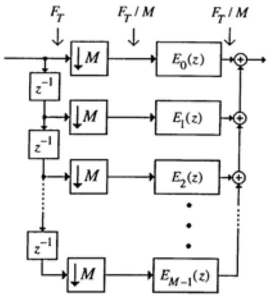


Computationally Efficient Interpolator and Decimator Structures

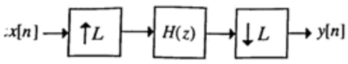
- Using the cascade equivalence #1 for decimation



- The structure reduces to a computationally more efficient structure:



Cascade Multirate Structure



- Expressing the transfer function H(z) in terms of its L-term Type I polyphase form:

$$H(z) = \sum_{k=0}^{L-1} z^{-k} E_k(z^L)$$

- The structure is equivalent to the time-invariant digital filter where $E_0(z)$ is the zeroth polyphase term