6 z-Transform

Introduction

- A generalization of the discrete-time Fourier transform leads to the z-transform, which is function of the complex variable z
- The use of z-transform techniques permits simple algebraic manipulations
- The z-transform has become an important tool in the analysis and design of digital filters
- The representation of an LTI discrete-time system in the z-domain is given by its transfer function which is the z-transform of the impulse response of the system

Introduction

- In this chapter, the alternate transform-domain representation of sequences and its properties is discussed
- The properties of the z-domain transfer function are studied
- The z-domain transfer function of the system is related to the frequency response of the system, which is the discrete-time Fourier transform of the impulse response
- The BIBO stability condition of an LTI system is derived in terms of its transfer function

Definition and Properties

- For a given sequence $g[n]$, its z-transform is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n]\ z^{-n}$$

where $z=\mathrm{Re}(z)+j\mathrm{Im}(z)$ is a continuous complex variable

The z-Transform

- The z-transform is often expressed as an operator indicated below

$$Z\{g[n]\} = G(z) = \sum_{n=-\infty}^{\infty} g[n]\ z^{-n}$$

- The operator $Z\{\cdot\}$ transforms a discrete-time sequence $g[n]$ into a function $G(z)$ of the complex variable z
- The relation can be expressed as

$$g[n] \overset{z}{\leftrightarrow} G(z)$$

The z-Transform

- Expressing the complex variable z in polar form $z = re^{j\omega}$, the definition of the z-transform reduces to

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]\ r^{-n} e^{-j\omega n}$$

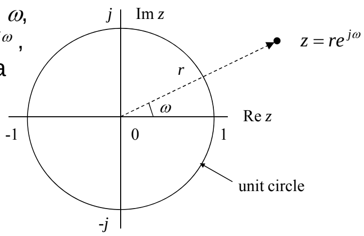
- Comparing the above equation with discrete-time Fourier transform $G(e^{j\omega})$ of the sequence $g[n]$

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]\ e^{-j\omega n}$$

i.e., $G(re^{j\omega})$ is the DTFT of the sequence $\{g[n]r^{-n}\}$

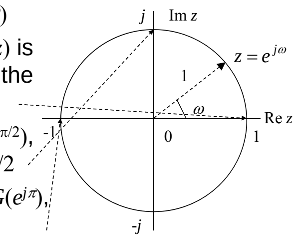
The z-Transform

- A geometrical interpretation of z-transform can be given by considering the location of the complex point z in the complex z-plane
- For fixed r and ω, the point $z = re^{j\omega}$, is at the tip of a vector
- The contour $|z|=1$ is the ***unit circle***



The z-Transform

- For $r=1$, i.e., $|z|=1$, the z-transform reduces to the Fourier transform $G(e^{j\omega})$
- At $z=1$, the value of $G(z)$ is $G(z)=G(1)=G(e^{j0})$, i.e., the value of $G(e^{j\omega})$ at $\omega=0$
- At $z=j$, $G(z)=G(j)=G(e^{j\pi/2})$, we have $G(e^{j\omega})$ at $\omega=\pi/2$
- At $z=-1$, $G(z)=G(-1)=G(e^{j\pi})$, we have $G(e^{j\omega})$ at $\omega=\pi$
- The frequency response $G(e^{j\omega})$ is obtained by evaluating $G(z)$ on the unit circle



Convergence of the z-Transform

- Like the discrete-time Fourier transform, there are conditions on the convergence of the infinite series expansion of the z-transform
- For a given sequence, the set $\mathcal{R}$ of values of z for which its z-transform converges is called the ***region of convergence*** (ROC)
- Without the knowledge of the ROC there is no unique relationship between the sequence and its z-transform
- **The z-transform must always be specified with its ROC !**

Convergence of the z-Transform

- From the interpretation of the z-transform $G(z)$ as the discrete-time Fourier transform of the sequence $g[n]r^{-n}$ it follows that the series of the z-transform definition converges if $g[n]r^{-n}$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} \left| g[n] \; r^{-n} \right| < \infty$$

- The sequence $g[n]r^{-n}$ can be made absolutely summable by choosing the value of r properly

Example: z-Transform of the Unit Step

- The z-transform of the unit step sequence

$$\mu(z) = \sum_{n=-\infty}^{\infty} \mu[n] \; z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$
$$= 1 + z^{-1} + z^{-2} + \ldots + z^{-n} + \ldots$$

which is a power series that converges to

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

- The region of convergence is the annular region in the z-plane $1 < |z| \leq \infty$

Example 6.1: z-Transform of a Causal Exponential Sequence

- The z-transform of the causal sequence $x[n]=\alpha^n \mu[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] \; z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$
$$= 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \ldots + \alpha^n z^{-n} + \ldots$$

- The above power series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |\alpha z^{-1}| < 1$$

- The region of convergence is the annular region in the z-plane $|z| > \alpha$, i.e., the outside of a circle with radius α

Example 6.2: z-Transform of an Anticausal Exponential Sequence

- The z-transform of the anti causal sequence $x[n] = -\alpha^n \mu[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^m = -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m$$

$$= -\frac{\alpha^{-1} z}{1-\alpha^{-1} z} = \frac{1}{1-\alpha z^{-1}}, \quad |\alpha^{-1} z| < 1$$

- Now, the region of convergence is the annular region in the z-plane $|z| < \alpha$

Region of Convergence of z-Transforms

Notice:

- The z-transforms of the two sequences $x[n]=\alpha^n\mu[n]$ and $x[n] = -\alpha^n\mu[-n-1]$ are identical even though the two parent sequences are different
- The only way a unique sequence can be associated with a z-transform is by specifying its ROC

Regions of Convergence: The DTFT and the z-Transform

- The DTFT, $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle
- The existence of the DTFT does not always imply the existence of the z-transform

Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n\mu[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_o n)\mu[n]$	$\frac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n \sin\omega_o n)\mu[n]$	$\frac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Rational z-Transforms

- LTI discrete-time systems have z-transforms which are rational functions of z^{-1}

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \ldots + p_{M-1} z^{M-1} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \ldots + d_{N-1} z^{N-1} + d_N z^{-N}}$$

- is a ratio of two polynomials $P(z)$ and $D(z)$
- The ***degree*** of the numerator polynomial $P(z)$ is M and that of the denominator polynomial $D(z)$ is N
- The degree of $H(z)$ is maximum of M and N

Rational z-Transforms

- An alternate representation of a rational z-transform is a ratio in positive powers of z

$$H(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \ldots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \ldots + d_{N-1} z + d_N}$$

- $H(z)$ can be factored into the form

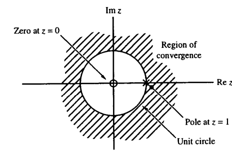
$$H(z) = \frac{p_0}{d_0} \frac{\prod_{l=1}^{M}\left(1-\xi_l z^{-1}\right)}{\prod_{l=1}^{N}\left(1-\lambda_l z^{-1}\right)} = \frac{p_0}{d_0} z^{N-M} \frac{\prod_{l=1}^{M}\left(z-\xi_l\right)}{\prod_{l=1}^{N}\left(z-\lambda_l\right)}$$

Zeroes and Poles

- At a root $z=\xi_l$ of the numerator polynomial, $H(\xi_l)=0$ and these values of z are called the ***zeroes*** of $H(z)$
- At a root $z=\lambda_l$ of the denominator polynomial, $H(\lambda_l) \rightarrow$ infinity and these values of z are called the ***poles*** of $H(z)$
- There are M finite zeroes and N finite poles of $H(z)$
- There are additional $(N\text{-}M)$ zeros at the origin if $N>M$ or $(N\text{-}M)$ poles at $z=0$ if $N<M$

Example: z-Transform of the Unit Step

- The region of convergence in the z-plane



$$\mu(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad \begin{cases} \text{zero}: & z=0 \\ \text{pole}: & z=1 \end{cases}$$

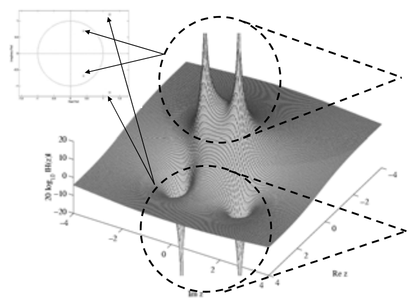
Rational z-Transforms

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|H(z)|$ of $H(z)$

$$H(z) = \frac{1-2.4z^{-1}+2.88z^{-2}}{1-0.8z^{-1}+0.64z^{-2}}$$

- The poles are at $z = 0.4 \pm j0.6928$ and the zeroes are at $z = 1.2 \pm j1.2$
- The 3-D plot is shown on next slide

Rational z-Transforms

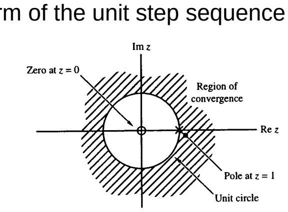


Regions of Convergence of a Rational z-Transform

- Recall the z-transform of the unit step sequence

$$\mu(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$\begin{cases} \text{zero}: & z=0 \\ \text{pole}: & z=1 \end{cases}$$



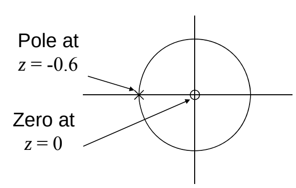
- The ROC of a rational z-transform is bounded by the locations of its poles

ROC of the z-Transform of a Causal Exponential Sequence

- Determine the ROC of the z-transform $H(z)$ of the causal sequence $h[n]=(-0.6)^n\mu[n]$

$$H(z) = \frac{1}{1+0.6z^{-1}} = \frac{z}{z+0.6}, \quad |z|>0.6$$

- The ROC is outside the circle going through the point z=-0.6 in the z-plane, extending to the infinity



Regions of Convergence

- Assume that $X(z)$ has simple poles at α and β with $|\alpha|<|\beta|$
- If $x[n]$ is right-sided sequence

$$x[n] = \left(r_1(\alpha)^n + r_2(\beta)^n\right)\mu[n-N_0]$$

where N_0 is an integer

- The z-transform of a right-sided sequence $(\gamma)^n\mu[n\text{-}N_0]$ exists if

$$\sum_{n=N_0}^{\infty} \left|(\gamma)^n z^{-n}\right| < \infty, \quad \text{for some } z$$

Regions of Convergence

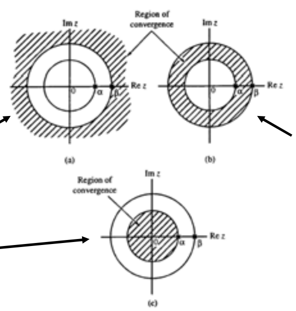
- The right-sided sequence $x[n]$ has thus a region of convergence (ROC) defined by $|\beta|<|z|\leq$ *infinity*, i.e., the ROC is bounded by the largest pole
- Similarly, a left-sided sequence has a ROC defined by $0|\leq|z|<|\alpha|$,i.e., the ROC is bounded by the smallest pole
- A two-sided sequence can be decomposed into a right-sided and left-sided sequence => The ROC is an annular region in the z-plane

Regions of Convergence

Right-sided sequence

Two-sided sequence

Left-sided sequence



The Inverse z-Transform

- For $z=re^{j\omega}$, the z-transform $G(z)$ is the Fourier transform of the exponentially weighted sequence $g[n]r^{-n}$
- The inverse Fourier transform

$$g[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega})e^{j\omega n}d\omega$$

- Changing the variable $z=re^{j\omega}$ gives the contour integral

$$g[n] = \frac{1}{2\pi j}\oint_{C'} G(z)z^{n-1}dz$$

where C' is a counterclockwise contour encircling the origin in the ROC of $G(z)$ gives the contour integral

The Inverse z-Transform

- The sequence $g[n]$ can be evaluated from its z-transform using the Cauchy's residue theorem

$$g[n] = \sum \left[\text{Residues of } G(z)z^{n\text{-}1} \text{ at the poles inside C}\right]$$

- Other simple methods for evaluating the inverse z-transform:
  - Partial fraction expansion of the rational $G(z)$
  - Long division of the numerator by the denominator of $G(z)$

z-Transform Properties

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n]+\beta h[n]$	$\alpha G(z)+\beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n-n_o]$	$z^{-n_o}G(z)$	$\mathcal{R}_g$, except possibly the point $z=0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\frac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z=0$ or ∞
Convolution	$g[n]\circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g\mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$		

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-}<|z|<R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-}<|z|<R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+}<|z|<1/R_{g^-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g^-}R_{h^-}<|z|<R_{g^+}R_{h^+}$.

Convolution Property

- The z-transform of the convolution is

$$g[n] \circledast h[n] \overset{z}{\leftrightarrow} G(z)H(z), \quad \text{ROC includes } R_g \cap R_h$$

- The z-transform of the convolution sum gives

$$Z\{g[n] \circledast h[n]\} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} g[k]h[n-k]\right)z^{-n} = \sum_{k=-\infty}^{\infty} g[k]\left(\sum_{n=-\infty}^{\infty} h[n-k]z^{-n}\right)$$

- Substituting $l=n-k$, then $n=l+k$, gives

$$Z\{g[n] \circledast h[n]\} = \sum_{k=-\infty}^{\infty} g[k]\left(\sum_{l=-\infty}^{\infty} h[l]z^{-(l+k)}\right) = \underbrace{\left(\sum_{k=-\infty}^{\infty} g[k]z^{-k}\right)}_{G(z)}\underbrace{\left(\sum_{l=-\infty}^{\infty} h[l]z^{-l}\right)}_{H(z)}$$

- By definition of the z-transform: $G(z)$ $H(z)$

Modulation Property

- The z-transform of the product sequence is

$$g[n]h[n] = \frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}dv, \quad \text{ROC includes } R_g R_h$$

where C is a closed counterclockwise contour encircling the origin in the common ROCs R_g and R_h

- For the ROCs $R_{g-}<|z|<R_{g+}$ and $R_{h-}<|z|<R_{h+}$ the ROC R_gR_h represents the region $R_{g-}R_{h-}<|z|<R_{g+}R_{h+}$
- The modulation theorem is also called the ***complex convolution theorem***

Modulation Property

- Substituting the inverse z-transform expression into the z-transform of the product sequence gives

$$Z\{g[n]h[n]\} = \sum_{n=-\infty}^{\infty} g[n]h[n]z^{-n} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi j}\oint_C G(v)v^{n-1}dv\right)h[n]z^{-n}$$

- Interchanging the order of integration and summation

$$Z\{g[n]h[n]\} = \frac{1}{2\pi j}\oint_C G(v)\left(\sum_{n=-\infty}^{\infty} h[n]\left(\frac{z}{v}\right)^{-n}\right)v^{-1}dv$$

$$= \frac{1}{2\pi j}\oint_C G(v)H\left(\frac{z}{v}\right)v^{-1}dv$$

Complex Convolution Theorem

- To see the similarity of the modulation property with the convolution, write v and z in polar form

$$v = \rho e^{j\theta} \quad \text{and} \quad z = re^{j\phi}$$

- The modulation property now becomes

$$Z\{g[n]h[n]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\rho e^{j\theta})H\left(\frac{re^{j\phi}}{\rho e^{j\theta}}\right)\rho^{-1}d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\rho e^{j\theta})H\left(\frac{r}{v}e^{j(\phi-\theta)}\right)d\theta$$

- This is often referred to as a ***periodic convolution***

The Transfer Function

- Generalization of $H(e^{j\omega})$ leads to the concept of transfer function

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

- In z-domain: $Y(z) = X(z)H(z)$
- Solving $H(z)$: $H(z) = \dfrac{Y(z)}{X(z)}$

  where $H(z) = \mathcal{Z}\{h[n]\} = \sum_{k=-\infty}^{\infty} h[k]z^{-n}$

$H(z)$ is called ***the transfer function*** or ***system function***

The Transfer Function

- If the region of convergence (ROC) of $H(z)$ includes the unit circle, the transfer function is related to the frequency response $H(e^{j\omega})$ of an LTI digital filter
- The frequency response is obtained by evaluating the z-transform on the unit circle, i.e.,

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- The frequency-domain behavior of a digital filter can be easily determined by graphical interpretation of $H(e^{j\omega})$

The Transfer Function Expressions

1) A Finite Impulse Response (FIR) digital filter
   - The impulse response is of finite length
   - The transfer function is a polynomial in z^{-1}
   - The realization is non-recursive

2) An Infinite Impulse Response (IIR) digital filter
   - The impulse response is of infinite length
   - The transfer function is a rational function in z^{-1}
   - The realization is recursive

FIR Digital Filters

$$h[n]=0;\quad n<N_1 \quad \text{and} \quad n>N_2$$

$$y[n]=\sum_{n=N_1}^{N_2} h[k]x[n-k]$$

$$Y(z)=\left(\sum_{n=N_1}^{N_2} h[n]z^{-n}\right)X(z)=H(z)X(z)$$

$$H(z)=\sum_{n=N_1}^{N_2} h[n]z^{-n}$$

FIR Digital Filters

$$\boxed{H(z)=\sum_{n=N_1}^{N_2} h[n]z^{-n}}$$

- For a causal FIR filter, N_1=0 and N_2>0
- All poles of an FIR filter are at the origin of the z-plane, and the ROC is the entire z-plane excluding the origin
- The transfer function is a polynomial in z^{-1}

IIR Digital Filters

- The input-output relation given by the difference equation

$$\sum_{k=0}^{N} d_k y[n-k]=\sum_{k=0}^{M} p_k x[n-k]$$

- Solving for $y[n]$

$$y[n]=\frac{1}{d_0}\sum_{k=0}^{M} p_k x[n-k]-\frac{1}{d_0}\sum_{k=1}^{N} d_k y[n-k]$$

- Output is obtained recursively from $x[n]$ and its previous M samples and N previous output samples

IIR Digital Filters

- Taking the z-transform of the difference equation

$$\left\{\sum_{k=0}^{N} d_k z^{-k}\right\}Y(z)=\left\{\sum_{k=0}^{M} p_k z^{-k}\right\}X(z)$$

- Solving for $H(z)$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}=\frac{p_0+p_1z^{-1}+p_2z^{-2}+\ldots+p_Mz^{-M}}{d_0+d_1z^{-1}+d_2z^{-2}+\ldots+d_Nz^{-N}}$$

- The transfer function is a rational function in z^{-1}

IIR Filters

- $H(z)$ can be written in the form

$$H(z)=z^{(N-M)}\frac{p_0z^M+p_1z^{M-1}+p_2z^{M-2}+\ldots+p_M}{d_0z^N+d_1z^{N-1}+d_2z^{N-2}+\ldots+d_N}$$

- Solving the roots of the numerator and denominator polynomials leads to the factored form of $H(z)$

$$H(z)=\frac{p_0}{d_0}\frac{\prod_{k=1}^{M}(1-\xi_k z^{-1})}{\prod_{k=1}^{N}(1-\lambda_k z^{-1})}=\frac{p_0}{d_0}z^{(N-M)}\frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

IIR Filters

$$H(z)=\frac{p_0+p_1z^{-1}+p_2z^{-2}+...+p_Mz^{-M}}{d_0+d_1z^{-1}+d_2z^{-2}+...+d_Nz^{-N}}=\frac{p_0}{d_0}\frac{\prod_{k=1}^{M}(1-\xi_k z^{-1})}{\prod_{k=1}^{N}(1-\lambda_k z^{-1})}$$

- The zeroes of $H(z)$ are $\{\xi_1,\xi_2,...,\xi_M\}$
- The poles of $H(z)$ are $\{\lambda_1,\lambda_2,...,\lambda_N\}$
- The coefficients p_k and d_k determine the locations of zeroes and poles, respectively

Evaluation of $H(e^{j\omega})$

$$H(e^{j\omega})=H_{re}(e^{j\omega})+H_{im}(e^{j\omega})=\left|H(e^{j\omega})\right|e^{j\arg[H(e^{j\omega})]}$$

- For a real coefficient transfer function

$$\left|H(e^{j\omega})\right|^2=H(e^{j\omega})H^*(e^{j\omega})=H(e^{j\omega})H(e^{-j\omega})$$
$$=H(z)H^*(z^{-1})\Big|_{z=e^{j\omega}}$$

- The values of the frequency response can be obtained by evaluating the z-transform on the unit circle in the z-plane, i.e., $H(e^{j\omega})$ is $H(z)$ at $z=e^{j\omega}$

Geometric Evaluation of $H(e^{j\omega})$

- The values of $H(e^{j\omega})$ can be estimated from the geometry of the pole/zero diagram

$$H(z)=\frac{p_0}{d_0}z^{(N-M)}\frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

- Writing the terms in polar form

$$(z-\xi_k)=B_ke^{j\theta_k},\quad (z-\lambda_k)=A_ke^{j\phi_k}$$
$$H(z)=z^{(N-M)}\frac{p_0\left(\prod_{k=1}^{M}B_k\right)e^{j\sum_{k=1}^{M}\theta_k}}{d_0\left(\prod_{k=1}^{N}A_k\right)e^{j\sum_{k=1}^{N}\phi_k}}$$

Geometric Evaluation of $H(e^{j\omega})$

- Each term $(z-\xi_k)$ and $(z-\lambda_k)$ can be interpreted as a vector in the z-plane with the magnitude, B_k and A_k, and the angle θ_k and ϕ_k
- Evaluating the "zero and pole vectors" on the unit circle gives the magnitude and phase responses of $H(e^{j\omega})$

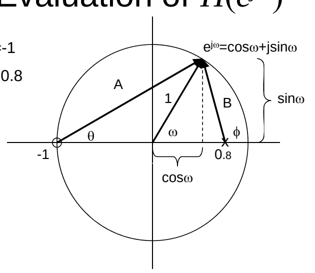
$$\left|H(e^{j\omega})\right|=\frac{p_0\prod_{k=1}^{M}B_k}{d_0\prod_{k=1}^{N}A_k}$$

$$\arg[H(e^{j\omega})]=\sum_{k=1}^{M}\theta_k-\sum_{k=1}^{N}\phi_k$$

Geometric Evaluation of $H(e^{j\omega})$

$$H(z)=\frac{z+1}{z-0.8}\quad\begin{cases}\text{Zero: } z=-1\\ \text{Pole: } z=0.8\end{cases}$$

$$A^2=(\cos\omega+1)^2+\sin^2\omega$$
$$B^2=(0.8-\cos\omega)^2+\sin^2\omega$$
$$\theta=\arctan\frac{\sin\omega}{\cos\omega+1}$$
$$\pi-\phi=\arctan\frac{\sin\omega}{0.8-\cos\omega}$$



$$\left|H(e^{j\omega})\right|=\left|\frac{e^{j\omega}+1}{e^{j\omega}-0.8}\right|=\left|\frac{\cos\omega+j\sin\omega+1}{\cos\omega+j\sin\omega-0.8}\right|=\left[\frac{(\cos\omega+1)^2+\sin^2\omega}{(\cos\omega-0.8)^2+\sin^2\omega}\right]^{1/2}$$

Stability Condition

- Bounded-input bounded-output (BIBO) stability: $h[n]$ is absolutely summable, i.e.,

$$S=\sum_{n=-\infty}^{\infty}\left|h[n]\right|<\infty$$

- The z-transform converges if $\sum_{n=-\infty}^{\infty}\left|h[n]z^{-n}\right|<\infty$ for which $h[n]r^{-n}$ is absolutely summable
- If the ROC includes the unit circle, then the digital filter is stable
- For a causal and stable digital filter the poles must be strictly inside the unit circle

Rational z-Transforms

- If the ROC includes the unit circle, the Fourier transform of the sequence can be obtained by evaluating the z-transform on the unit circle
- In addition, the ROC of the z-transform of the impulse response of a causal LTI system is related to the BIBO stability of the system

The ROC of the z-transform of a causal and stable discrete-time system includes the unit circle and the infinity in the z-plane

Summary

- Analysis equation converts from time-domain representation to transform-domain representation
- Synthesis equation is used for the reverse process
- Important and useful characterization of an LTI discrete-time system is its transfer function given by the z-transform of its impulse response
- The behavior of the system is determined by the transfer function and its poles and zeros
- Stability of the system is determined by the pole locations