Tik-61.3030 Principles of Neural Computing

Raivio, Venna

Exercise 10

1. The thin-plate-spline function is described by

$$\varphi(r) = \left(\frac{r}{\sigma}\right)^2 \log\left(\frac{r}{\sigma}\right) \quad \text{for some } \sigma > 0 \text{ and } r \in \mathbb{R}^+$$

   Justify the use of this function as a translationally and rotationally invariant Green's function. Plot the function graphically. (Haykin, Problem 5.1)

2. The set of values given in Section 5.8 for the weight vector $\mathbf{w}$ of the RBF network in Figure 5.6. presents one possible solution for the XOR problem. Solve the same problem by setting the centers of the radial-basis functions to

$$\mathbf{t}_1 = [-1, 1]^T \text{ and } \mathbf{t}_2 = [1, -1]^T.$$

   (Haykin, Problem 5.2)

3. Consider the cost functional

$$\mathcal{E}(F^*) = \sum_{i=1}^{N} \left[d_i - \sum_{j=1}^{m_1} w_j G(\|\mathbf{x}_j - \mathbf{t}_i\|) \right]^2 + \lambda \|\mathbf{D}F^*\|^2$$

   which refers to the approximating function

$$F^*(\mathbf{x}) = \sum_{i=1}^{m_1} w_i G(\|\mathbf{x} - \mathbf{t}_i\|).$$

   Show that the cost functional $\mathcal{E}(F^*)$ is minimized when

$$(\mathbf{G}^T\mathbf{G} + \lambda\mathbf{G}_0)\mathbf{w} = \mathbf{G}^T\mathbf{d}$$

   where the N-by-m_1 matrix $\mathbf{G}$, the m_1-by-m_1 matrix $\mathbf{G}_0$, the m_1-by-1 vector $\mathbf{w}$, and the N-by-1 vector $\mathbf{d}$ are defined by Equations (5.72), (5.75), (5.73), and (5.46), respectively. (Haykin, Problem 5.5)

4. Consider more closely the properties of the singular-value decomposition (SVD) discussed very briefly in Haykin, p. 300.

   (a) Express the matrix $\mathbf{G}$ in terms of its singular values and vectors.

   (b) Show that the pseudoinverse $\mathbf{G}^+$ of $\mathbf{G}$ can be computed from Equation (5.152):

$$\mathbf{G}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^T.$$

   (c) Show that the left and right singular vector $\mathbf{u}_i$ and $\mathbf{v}_j$ are obtained as eigenvectors of the matrices $\mathbf{G}\mathbf{G}^T$ and $\mathbf{G}^T\mathbf{G}$, respectively, and the squared singular values are the corresponding nonzero eigenvalues.