T-61.140 Formulae

$x(t)$ refers to continuous-time signal and $x[n]$ discrete-time sequence. For signals and systems in general, y is output, h is impulse response of the system and x input.

$\delta(t)$ and $\delta[n]$ are unit impulse functions.
$u(t)$ and $u[n]$ are unit step functions.
$s(t)$ and $s[n]$ are step response functions.
Rise time is the period when step response increases from 10% up to 90% of its maximum value.

Fourier series: ω_0 is fundamental (basic) angular frequency, f_0 fundamental frequency, T and N lengths of period. For continuous $\omega_0 = 2\pi f_0 = 2\pi/T$, $T = \frac{1}{f_0}$, and for discrete ($N \in \mathbb{Z}$) $\omega_0 = 2\pi f_0 = 2\pi/N$, $N = \frac{1}{f_0}$.

Complex numbers, Euler, geometric series, basic functions:

$$
\begin{aligned}
z &= x + jy = re^{j\theta} \\
e^{j\theta} &= \cos(\theta) + j\,\sin(\theta)
\end{aligned}
$$

$\cos(0) = 1,\ \cos(\pi/4) = 1/\sqrt{2} \approx 0.71,\ \cos(\pi/2) = 0,\ \cos(\pi) = -1$
$\sin(0) = 0,\ \sin(\pi/4) = 1/\sqrt{2} \approx 0.71,\ \sin(\pi/2) = 1,\ \sin(\pi) = 0$

$$
\begin{aligned}
\sum_{n=0}^{+\infty} a^n &= \frac{1}{1-a}, |a| < 1 \\
\sum_{n=0}^{N-1} a^n &= \frac{1-a^N}{1-a}, |a| < 1
\end{aligned}
$$

$$
\begin{aligned}
u(t) &= \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} = \int_{-\infty}^{\tau} \delta(\tau) d\tau \\
\delta[n] &= \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \\
u[n] &= \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \\
s(t) &= u(t) * h(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \\
s[n] &= u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]
\end{aligned}
$$

Computation of Dirac delta function:

$$
\int_{-\infty}^{\infty} \delta(t - t_0)\ x(t)\ dt = x(t_0)
$$

Convolution:

$$\begin{aligned} y(t) &= x(t) * h(t) = h(t) * x(t) = \int_{\tau=-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau \\ y[n] &= x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k] \end{aligned}$$

FOURIER SERIES

Fourier series of continuous-time periodic signal:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{jk\omega_0 t} \qquad \text{where} \quad a_k = \frac{1}{T} \int_T x(t) \, e^{-jk\omega_0 t} dt$$

Properties of continuous-time Fourier-series:

Here $x(t)$ and $y(t)$ are periodic with period T, and a_k and b_k are corresponding Fourier series coefficients.

- linearity $Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k$
- time shifting $x(t - t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$
- frequency shifting $e^{jM\omega_0} x(t) \leftrightarrow a_{k-M}$

Fourier series of discrete-time periodic signal:

$$x[n] = \sum_{k=\langle N \rangle} a_k \, e^{jk\omega_0 n} \qquad \text{where} \quad a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \, e^{-jk\omega_0 n}$$

Properties of discrete-time Fourier-series:

Here $x[n]$ and $y[n]$ are periodic with period N, and a_k and b_k are corresponding Fourier series coefficients with period N.

- linearity $Ax[n] + By[n] \leftrightarrow Aa_k + Bb_k$
- time shifting $x[n - n_0] \leftrightarrow a_k e^{-jk(2\pi/N)n_0}$
- frequency shifting $e^{jM(2\pi/N)n} x[n] \leftrightarrow a_{k-M}$

CONT. FOURIER TRANSFORM (CTFT)

Fourier transform of continuous-time aperiodic signal:

$$F^{-1}\{X(j\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t} d\omega$$
$$F\{x(t)\} = X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

Properties of oontinuous-time Fourier transform:
Here $x(t) \leftrightarrow X(j\omega)$ and $y(t) \leftrightarrow Y(j\omega)$.

- linearity $ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$
- time shifting $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$
- frequency shifting $e^{j\omega_0 t} x(t) \leftrightarrow X(j(\omega - \omega_0))$
- convolution $x(t) * y(t) \leftrightarrow X(j\omega) \cdot Y(j\omega)$
- multiplication $x(t) \cdot y(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta) Y(j(\omega - \theta)) d\theta$
- derivation in time $\frac{d}{dt} x(t) \leftrightarrow j\omega X(j\omega)$

Transform pairs of oontinuous-time Fourier transform:
Here signal $\leftrightarrow$ F-transform.

- $\delta(t) \leftrightarrow 1$
- $\delta(t - t_0) \leftrightarrow e^{-j\omega t_0}$
- $e^{-at} u(t) \leftrightarrow \frac{1}{a + j\omega}$
- $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \quad \leftrightarrow \frac{2\sin(\omega T_1)}{\omega}$
- $\frac{\sin(Wt)}{\pi t} \leftrightarrow X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$

DISCR.TIME FOURIER TRANSFORM (DTFT)

Discrete-time Fourier-transform of aperiodic signal

$$F^{-1}\{X(e^{j\omega})\} = x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

$$F\{x[n]\} = X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

Properties of Discrete-time Fourier-transform:

Here $x[n] \leftrightarrow X(e^{j\omega})$ and $y[n] \leftrightarrow Y(e^{j\omega})$, and $X(e^{j\omega})$ and $Y(e^{j\omega})$ are 2π-periodic.

- linearity $ax[n] + by[n] \leftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$
- time shift $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega})$
- frequency shift $e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
- convolution $x[n] * y[n] \leftrightarrow X(e^{j\omega}) \cdot Y(e^{j\omega})$
- multiplication $x[n] \cdot y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$
- difference in time $x[n] - x[n-1] \leftrightarrow (1 - e^{j\omega}) X(e^{j\omega})$

Discrete-time Fourier-transform pairs:

Here's signal $\leftrightarrow$ F-transform.

- $\sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - (2\pi/N)k)$
- $\cos(\omega_0 n) \leftrightarrow \pi \sum_{l=-\infty}^{+\infty} \Big(\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l) \Big)$
- $\sin(\omega_0 n) \leftrightarrow (\pi/j) \sum_{l=-\infty}^{+\infty} \Big(\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l) \Big)$
- $\delta[n] \leftrightarrow 1$
- $\delta[n - n_0] \leftrightarrow e^{-j\omega n_0}$
- $a^n u[n] \leftrightarrow \frac{1}{1 - a\, e^{-j\omega}}$
- $x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases} \quad \leftrightarrow \frac{\sin(\omega(N_1 + 0.5))}{\sin(0.5\omega)}$
- $\frac{\sin(Wn)}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right), 0 < W < \pi \leftrightarrow X(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| < W \\ 0, & W < |\omega| \le \pi \end{cases}$